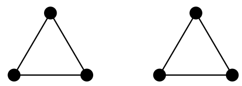
The University of Sydney

MATH2969/2069

Graph Theory **Tutorial 5 (Week 12) Solutions** 2008

1. (*i*) Let G be the disconnected planar graph shown.
Draw its dual G^*, and the dual of the dual $(G^*)^*$.

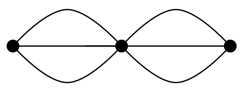
(*ii*) Show that if G is a disconnected planar graph, then G^* is connected.
Deduce that $(G^*)^*$ is not isomorphic to G.



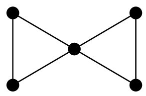
G

Solution.

(*i*)



G^*



$(G^*)^*$

(*ii*) A connected graph is one in which there exists a path between any pair of vertices.

Let G be a disconnected planar graph, and consider its dual G^*. In G^*, there is clearly a path between any two vertices which correspond to faces within one of the components of G. Let v be the vertex in G^* corresponding to the infinite face of G. Then v is adjacent to at least one vertex (corresponding to a face) in each component of G. So if u and w are two vertices of G^* which correspond to faces in different components of G, then there is a path from u to w, via v. Hence G^* is connected.

Since G^* is connected, its dual $(G^*)^*$ is also connected. But G is disconnected, and so $(G^*)^*$ is not isomorphic to G.

2. A certain polyhedron has faces which are triangles and pentagons, with each triangle surrounded by pentagons and each pentagon surrounded by triangles. If every vertex has the same degree, p say, show that $\frac{1}{e} = \frac{1}{p} - \frac{7}{30}$. Deduce that $p = 4$, and that there are 20 triangles and 12 pentagons. Can you construct such a polyhedron?

State the dual result.

Solution.

Let T be the number of triangles, and P the number of pentagons. Then the number of faces, $f = T + P$.

Since each triangle is bounded by 3 edges, and each pentagon by 5, we have $3T + 5P = 2e$.

But each edge adjoins exactly one triangle and exactly one pentagon, so $3T = 5P$. Hence,

$$T = \frac{2e}{6} = \frac{e}{3}, \quad P = \frac{2e}{10} = \frac{e}{5}, \quad f = P + T = \frac{e}{3} + \frac{e}{5} = \frac{8e}{15}.$$

Also, assuming every vertex has degree p, we have $pv = 2e$, or $v = 2e/p$. Now substitute into Euler's formula:

$$\frac{2e}{p} - e + \frac{8e}{15} = 2.$$

Divide by $2e$ and simplify:

$$\frac{1}{e} = \frac{1}{p} - \frac{7}{30}$$

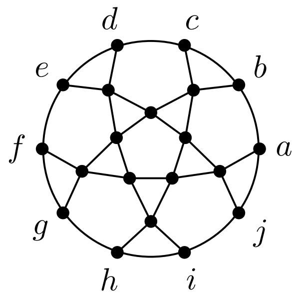
Clearly, $\frac{1}{e}$ must be a positive number, so we must have $\frac{1}{p} - \frac{7}{30} \geq 0$, or $p \leq 30/7$. But p must be an integer ≥ 3. So the only possible values for p are 3 or 4.

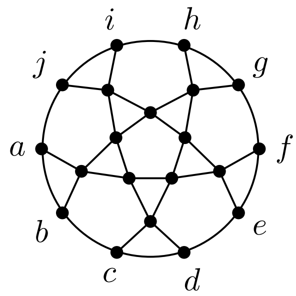
If $p = 3$, $e = 10$, but then $v = 2e/p = 20/3$, which is not an integer.

(Actually, it is easy enough to see that it is not possible to put together triangles and pentagons in the way specified, such that the degree of each vertex is three. Try drawing it and see!)

If $p = 4$, $e = 60$ and $T = 20$ and $P = 12$, as required.

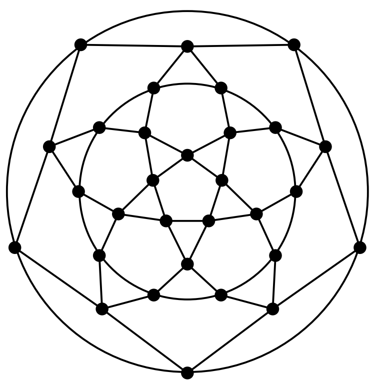
Such a polyhedron may be constructed in two halves:





We then stitch together along the edges which correspond in the labeling.

It is certainly possible to draw the full graph, with a little more effort:



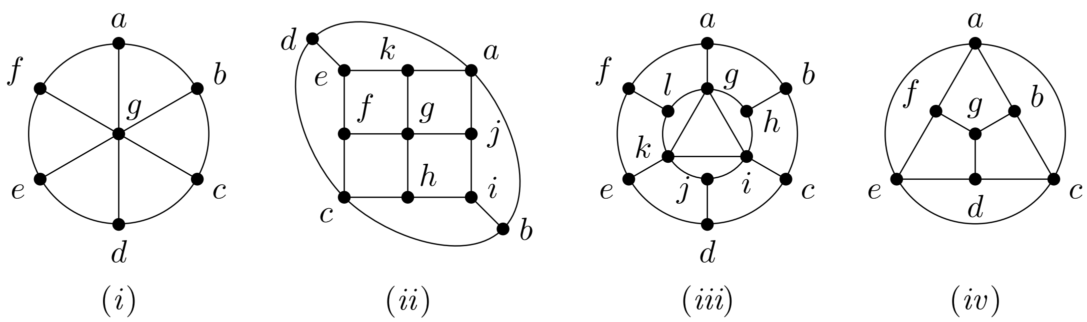
Note that the outside region is one of the pentagons.

The dual result is obtained by replacing vertices by faces and faces by vertices: "A certain polyhedron has vertices which have degrees 3 and 5, with each vertex of degree 3 adjacent to 3 vertices of degree 5, and each vertex of degree

5 adjacent to 5 vertices of degree 3.
If every face is bounded by the same number of edges, then that number is 4, and there are 20 vertices of degree 3 and 12 vertices of degree 5."

(Each vertex in the dual graph corresponds to a region in the original graph, and an edge incident to two adjacent regions in the original is an edge in the dual joining the corresponding vertices of the dual.)

3. Determine the chromatic number of each of the following graphs:



Solution.

A graph G is k-colourable if its vertices can be coloured using k colours in such a way that no two adjacent vertices have the same colour. The chromatic number of a simple graph G, written $\chi(G)$, is defined to be the smallest integer k for which G is k-colourable.

Useful facts: If G contains (a subgraph isomorphic to) K_n, then $\chi(G) \geq n$. If G contains an odd circuit, then $\chi(G) \geq 3$. If G has at least two vertices and no odd circuits, then $\chi(G) = 2$.

(*i*) This graph contains several triangles, so $\chi(G) \geq 3$. On the other hand we can easily find a 3-colouring (eg., a, c, e red; b, d, f white; g blue), so $\chi(G) = 3$.

(*ii*) The graph has no odd cycles, so $\chi(G) = 2$. (A 2-colouring is easily found (eg., a, c, e, g, i white, b, d, f, h, j, k black).

(*iii*) Since there are triangles, $\chi(G) \geq 3$. We can find a 3-colouring (eg., a, d, h, k red; b, e, i, l white; c, f, g, j blue), so $\chi(G) = 3$.

(*iv*) Since there are triangles, (eg., $\{a, b, c\}$), $\chi(G) \geq 3$, but is G 3-colourable? If so, without loss of generality let a, b, c be red, white, blue respectively.
Then e, adjacent to both a and c, must be white.
Also f, adjacent to both a and e, must be blue.
Also d, adjacent to both c and e, must be red.
Then, however, a fourth colour is needed for g, which is adjacent to b, d and f.

Hence $\chi(G) = 4$.

4. For each of the following graphs, what does Brooks' Theorem tell you about the chromatic number of the graph? Find the chromatic number of each graph.

(*i*) The complete graph K_{20}.
(*ii*) The bipartite graph $K_{10,20}$.
(*iii*) A cycle with 20 edges.
(*iv*) A cycle with 29 edges.

(*v*) The cube graph Q_3. (*vi*) The dual of Q_3.

Solution.

Let $\Delta(G)$ be the maximum of the degrees of the vertices of a graph G. Brooks' Theorem states that, for all graphs other than complete graphs or odd cycles, the chromatic number of a graph G, $\chi(G) \leq \Delta(G)$. For complete graphs, and for odd cycles, $\chi(G) = 1 + \Delta(G)$.

(*i*) $\Delta(K_{20}) = 19$, $\chi(K_{20}) = 1 + 19 = 20$. (Clearly, each of the 20 vertices must be coloured differently for a proper colouring.)

(*ii*) $\Delta(K_{10,20}) = 20$, so Brooks' Theorem says that $\chi(K_{10,20}) \leq 20$. Of course, the chromatic number of any bipartite graph is 2, so $\chi(K_{10,20}) = 2$.

(*iii*) In a cycle, all vertices have degree 2. By Brooks' Theorem, the chromatic number of a cycle with an even number of edges is at most 2, and clearly 2 colours are required, so the chromatic number is equal to 2.

(*iv*) By Brooks' Theorem, the chromatic number of a cycle with an odd number of edges is 3.

(*v*) Q_3 is regular of degree 3, and so by Brooks' Theorem $\chi(Q_3) \leq 3$. In fact, Q_3 is a bipartite graph with chromatic number 2.

(*vi*) The dual of Q_3 is the octahedron, in which every vertex has degree 4, so by Brooks' Theorem its chromatic number is at most 4. Note that the graph contains triangles, so its chromatic number is at least 3. It is easy to find a 3-colouring, and so $\chi(\text{the octahedron}) = 3$.

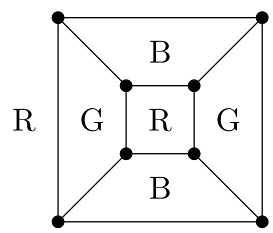
5. (*i*) Determine the minimum number of colours required to colour the faces of Q_3 in such a way that adjoining faces have a different colour.

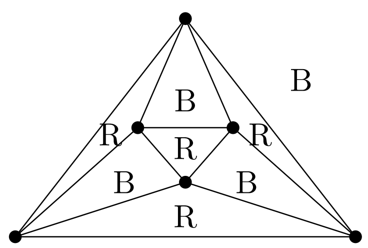
(*ii*) Repeat part (*i*) for the dual of Q_3.

Solution.

The graph Q_3 is the polyhedral graph corresponding to a cube. It is clear that at least 3 colours are needed to colour the faces of a cube so that no two adjoining faces have the same colour. The diagram below indicates one way in which to colour the faces of the cube using three colours (where R=red, B=blue, G=green).

The octahedron is the dual of the cube. For an octahedron only two colours are necessary. The diagram shows a 2-colouring of the faces of an octahedron.





Note that the minimum number of colours required to colour the faces of the cube is equal to the chromatic number of the octahedron, and vice versa.

6. Show that a simple connected planar graph with 17 edges and 10 vertices cannot be properly coloured with two colours.
(Hint: Show that such a graph must contain a triangle.)

Solution.

Suppose such a graph had no triangles. Then, since it is not a tree, each face must be bounded by at least 4 edges, and so $4f \leq 2e$, or $2f \leq e$. However, by Euler's formula, $f = 2 - v + e = 2 - 10 + 17 = 9$, so $2f = 2 \times 9 > 17 = e$ – a contradiction.

Hence the graph has at least one triangle and so is not 2-colourable.

7. Let T be a tree with at least 2 vertices. Prove that $\chi(T)$=2.

Solution.

Note that we need at least two colours to properly colour T, so if we prove that T is 2-colourable then $\chi(T)$=2.
Use induction on the number of vertices.
If there are 2 vertices, then the tree is clearly 2-colourable.
Suppose that a tree with k vertices is 2-colourable.
Let T be a tree with $k+1$ vertices, and remove from T one of its vertices with degree 1. (Every tree has at least two vertices with degree 1.) This leaves a tree with k vertices, which is 2-colourable by the induction hypothesis. Colour the smaller tree with 2 colours, restore the removed vertex and colour it with the colour not used on the (one) vertex to which it is adjacent. Hence T is 2-colourable, and the result follows.

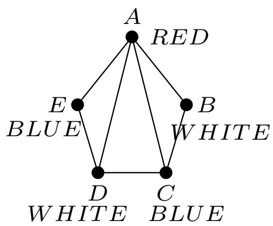
8. Hubert keeps five varieties (A, B, C, D, E) of snakes in boxes in his apartment. Some varieties attack other varieties, and can't be kept together. In the table, an asterisk indicates that varieties can't be kept together. What is the minimum number of boxes needed?

	A	B	C	D	E
A	–	*	*	*	*
B	*	–	*	–	–
C	*	*	–	*	–
D	*	–	*	–	*
E	*	–	–	*	–

Solution.

Construct a graph with varieties of snakes as vertices, and edges joining varieties which cannot be kept together. Then the minimum number of boxes required is the chromatic number of the graph.

Since the graph contains triangles, at least 3 colours are needed. A 3-colouring can be found, as shown. Hence the minimum number of boxes required is three.



9. Determine the number of ways in which each of the following graphs can be properly coloured, given λ different colours.

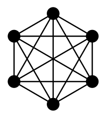
(i) The complete graph K_6. (ii) The star graph $K_{1,5}$.

(iii) The linear graph L_6.

Solution.

Recall that a colouring of a graph is an assignment of a colour (from some set of available colours) to each vertex, with the condition that no two adjacent vertices may receive the same colour.

Recall also that the chromatic polynomial of a graph G, written $P_G(\lambda)$, is the function in the one integer variable λ, such that if λ is any integer ≥ 0, $P_G(\lambda)$ is the number of ways of colouring G if λ colours are available. The function $P_G(\lambda)$ is a polynomial with integer coefficients.

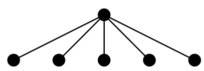
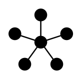
(i) $G = K_6 =$ 

With λ colours available, and starting with no vertices coloured, colour any vertex (λ choices), then another vertex (only $\lambda-1$ choices, the second vertex being adjacent to the first), then another (only $\lambda - 2$ choices, the third vertex being adjacent to both the previous), etc., there being $\lambda - 5$ ways of colouring the final vertex. (Note that at each step the number of choices does not depend on the previously made choices.)

By the product principle, the total number of colourings is therefore

$$P_G(\lambda) = \lambda_{(6)} = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5).$$

(ii)

$G = K_{1,5} =$  $=$ 

First colour the vertex of degree 5 (λ choices), then each of the other 5 vertices ($\lambda - 1$ choices each, as each is adjacent to just one already coloured vertex). Hence

$$P_G(\lambda) = \lambda(\lambda-1)^5.$$

(Of course, $K_{1,5}$ is a tree with 6 vertices, and if T is a tree with n vertices then $P_T(\lambda) = \lambda(\lambda-1)^{n-1}$.)

(iii) $G = L_6 =$

This is another tree on 6 vertices, and so the polynomial is as in part *(ii)*. Or one can proceed as follows: Colour the vertices in order from the left, starting with a degree 1 vertex (λ choices), and continuing with an adjacent vertex, etc. ($\lambda - 1$ choices each). Hence

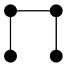
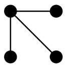
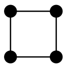
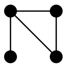
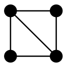
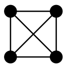
$$P_G(\lambda) = \lambda(\lambda-1)^5.$$

Note that taking the vertices in a carefully chosen order was important here. Starting with the two degree 1 vertices, for example (λ choices each), would eventually land us in difficulties, as the last vertex to be coloured would be adjacent to 2 already coloured vertices, and the number of choices would depend on whether they had been coloured the same or differently.

10. (*i*) Find the chromatic polynomials of each of the six connected simple graphs on four vertices.

(*ii*) Verify that each of the polynomials in (*i*) has the form $\lambda^4 - e\lambda^3 + a\lambda^2 - b\lambda$ where e is the number of edges and a and b are positive constants.

Solution.

G		$P_G(\lambda)$
	$= L_4$	$\lambda(\lambda-1)^3 = \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$
	$= K_{1,3}$	$\lambda(\lambda-1)^3 = \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$
	$= C_4$	$\lambda(\lambda-1)(\lambda^2 - 3\lambda + 3) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$
		$\lambda(\lambda-1)^2(\lambda-2) = \lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda$
		$\lambda(\lambda-1)(\lambda-2)^2 = \lambda^4 - 5\lambda + 8\lambda^2 - 4\lambda$
	$= K_4$	$\lambda(\lambda-1)(\lambda-2)(\lambda-3) = \lambda^4 - 6\lambda^3 + 12\lambda^2 - 6\lambda$

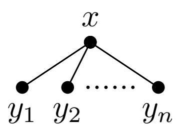
All the polynomials are easily found, except that for C_4. As we saw in lectures, the polynomial for C_4 can be found by adding together the number of ways of colouring C_4 in which a pair of non-adjacent vertices are coloured the same, and the number of ways in which that pair of vertices is coloured differently. This method gives

$$P_{C_4}(\lambda) = \lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)^2.$$

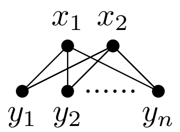
The verifications required in (*ii*) are by inspection.

11. Find the chromatic polynomials of $K_{1,n}$, $K_{2,n}$ and $K_{3,n}$.

Solution.

$G = K_{1,n} =$ 

The graph is a tree on $(n+1)$ vertices, so $P_G(\lambda) = \lambda(\lambda-1)^n$.

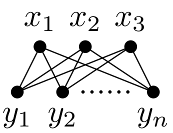
$G = K_{2,n} =$ 

Colour x_1 and x_2 first, but then, to colour $y_1, y_2, \ldots, y_n$, we need to consider two mutually exclusive cases:

Case 1: x_1 and x_2 have been coloured differently: λ choices for x_1 then $\lambda - 1$ choices for x_2. In this case each of $y_1, y_2, \ldots, y_n$ may then be coloured in $\lambda - 2$ ways: $\lambda(\lambda-1)(\lambda-2)^n$ ways all together.

Case 2: x_1 and x_2 have been given the same colour: λ choices for that colour. In this case each of y_1, y_2, ... , y_n may be coloured in $\lambda - 1$ ways: $\lambda(\lambda-1)^n$ ways all together. Using the addition principle for combining mutually exclusive cases, we get:

$$P_G(\lambda) = \lambda(\lambda-1)(\lambda-2)^n + \lambda(\lambda-1)^n.$$

$G = K_{3,n} =$ 

Here, if colouring x_1, x_2, x_3 first, there are three mutually exclusive cases to consider:

Case 1: All of x_1, x_2, x_3 have been given different colours: λ choices for x_1, $\lambda - 1$ for x_2 and $\lambda - 2$ for x_3. In this case there are $\lambda - 3$ choices for each of y_1, y_2, ... , y_n: $\lambda(\lambda-1)(\lambda-2)(\lambda-3)^n$ ways all together.

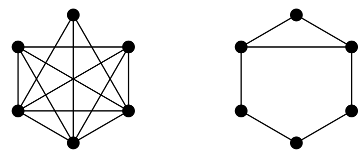
Case 2: Two of x_1, x_2, x_3 have been given the same colour, the other a different colour. Note that there are $\binom{3}{2} = 3$ choices for which two have the same colour, λ for what colour that is, then $\lambda - 1$ for the different colour. In this case there are $\lambda - 2$ choices for each of y_1, y_2, ... , y_n: $3\lambda(\lambda-1)(\lambda-2)^n$ ways all together.

Case 3: All of x_1, x_2, x_3 have been given the same colour: λ choices for that colour. In this case there are $\lambda-1$ choices for each of y_1, y_2, ... , y_n: $\lambda(\lambda-1)^n$ ways all together.

Combining these by the addition principle:

$$P_G(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)^n + 3\lambda(\lambda-1)(\lambda-2)^n + \lambda(\lambda-1)^n.$$

12. State two reduction formulas for chromatic polynomials. Use whichever seems appropriate to calculate the chromatic polynomial for each of the two given graphs. Also determine the chromatic number of each graph.



Solution.

For a simple graph G, the two forms are

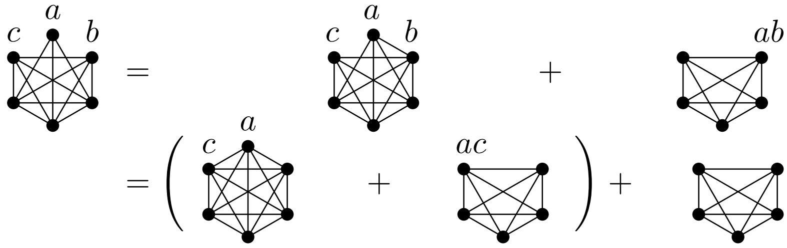
$$P_G(\lambda) = P_{G+e}(\lambda) + P_{G|e}(\lambda) \tag{1}$$
$$P_G(\lambda) = P_{G-e}(\lambda) - P_{G|e}(\lambda) \tag{2}$$

where, in (1), $G + e$ is G with a new edge e added (connecting two vertices which are non-adjacent in G), and, in (2), $G - e$ is G with an existing edge e removed. In each case $G|e$ is G "contracted along e", i.e., is the result of identifying the two vertices joined by e, a and b say, as a single vertex ab. Any vertex adjacent to either a or b or both in G is adjacent to ab in $G|e$, and all other adjacencies are preserved.

We know the chromatic polynomials of the complete graph K_n, and of any n-vertex tree T_n: $\lambda(\lambda-1)\cdots(\lambda-n+1)$ and $\lambda(\lambda-1)^{n-1}$ respectively. So it is appropriate to use (1) on "nearly complete" graphs, and (2) on graphs with few circuits, as in each case only a few applications are needed to reduce to sums/differences of known polynomials.

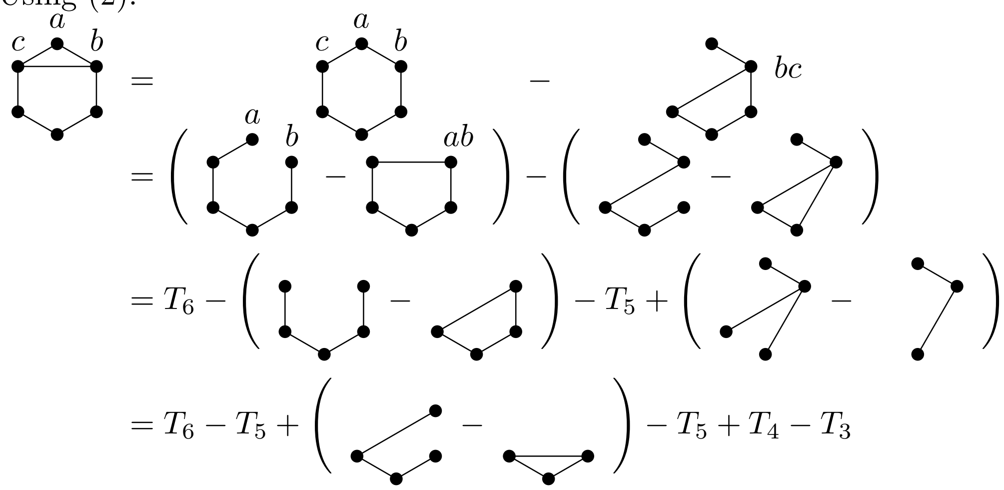
Note that we are using each graph diagram to mean the chromatic polynomial of the graph in the following.

Using (1):



$$\begin{aligned}
&= K_6 + K_5 + K_5\\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)\\
&\qquad + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)\\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3)^2(\lambda-4).
\end{aligned}$$

Using (2):



$$\begin{aligned}
&= T_6 - 2T_5 + 2T_4 - T_3 - K_3\\
&= \lambda(\lambda-1)^5 - 2\lambda(\lambda-1)^4 + 2\lambda(\lambda-1)^3 - \lambda(\lambda-1)^2 - \lambda(\lambda-1)(\lambda-2)\\
&= \text{(eventually) } \lambda(\lambda-1)(\lambda-2)^2(\lambda^2-2\lambda+2).
\end{aligned}$$

Recall that the chromatic number of a graph is the minimum number of colours with which the vertices of the graph may be properly coloured (i.e., with no pair of adjacent vertices having the same colour).

The first graph contains K_5 as a subgraph, so needs 5 colours. Inspection shows a sixth colour is not needed. Hence the chromatic number is 5. (Alternatively, observe that 5 is the first positive integer which is not a zero of the chromatic polynomial.)

The second graph contains odd length circuits, so needs 3 colours. Inspection shows a 4th colour is not needed. Hence the chromatic number is 3. (Alternatively, observe that 3 is the first positive integer which is not a zero of the chromatic polynomial.)

13. Find the chromatic polynomial of C_5, the cycle with 5 vertices.

Solution.

Using the formula $P_G(\lambda) = P_{G-e}(\lambda) - P_{G|e}(\lambda)$ on C_5, we obtain

$$\begin{aligned} P_{C_5}(\lambda) = P_{T_5}(\lambda) - P_{C_4}(\lambda) &= P_{T_5}(\lambda) - [P_{T_4}(\lambda) - P_{C_3}(\lambda)] \\ &= \lambda(\lambda-1)^4 - \lambda(\lambda-1)^3 + \lambda(\lambda-1)(\lambda-2) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 2\lambda + 2) \end{aligned}$$

(T_n denotes a tree on n vertices.)

14. Explain why the chromatic polynomial $P_G(\lambda)$ of a planar graph G cannot contain a term $(\lambda - k)$ for any $k \geq 4$.

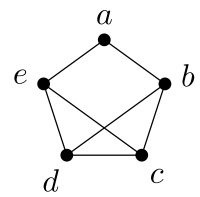
Solution.

If $P_G(\lambda)$ contains the term $(\lambda - k)$, then $P_G(k) = 0$, and there are zero ways to colour the graph with k colours. In other words, the graph is not k-colourable.

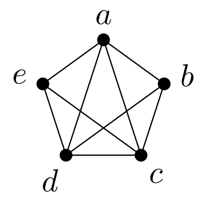
Since any planar graph is 4-colourable, and therefore k-colourable for all $k \geq 4$, its chromatic polynomial cannot contain a term $(\lambda - k)$ for any $k \geq 4$.

15. Find the chromatic index (or edge-chromatic number) of the graph G, where G is:

(a)



(b)



Solution.

Recall that a graph G is properly edge-coloured when its edges are assigned colours and no two edges incident at the same vertex have the same colour. Recall also that the chromatic index of G (also called the edge-chromatic number of G), $\chi'(G)$, is the minimum number of colours required to edge-colour G properly.

A theorem of Vizing states that for a simple graph G, $\chi'(G)$ is either Δ or $\Delta + 1$, where Δ is the maximum vertex degree of G. To determine which of these two is correct, we might find a way of edge-colouring G properly with just Δ colours, or prove it is impossible with just Δ colours.

(a) Here G has maximum degree 3 (all vertices having degree 3 except a, which has degree 2). So obviously at least 3 colours are needed to edge colour G properly (eg., the three edges incident to b require three different colours). By Vizing's theorem, $\chi'(G)$ is either 3 or 4.

Try to colour G properly with just 3 colours: R, G and B, say.

Without loss of generalty, $[a, b] = R$, $[b, c] = G$ and $[b, d] = B$.

Hence $[c, d] = R$, so $[c, e] = B$ and $[d, e] = G$.

Now a fourth colour is needed to colour $[a, e]$ properly. Hence $\chi'(G) = 4$.

(b) Vizing's theorem here shows $\chi'(G)$ is either 4 (the maximum vertex degree) or 5. Try using just 4 colours: R, B, G and Y, say.

Without loss of generalty, $[a, b] = R$, $[a, c] = G$, $[a, d] = B$ and $[a, e] = Y$.

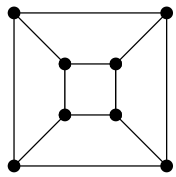
Then $[c, d]$ cannot be G or B, so is either R or Y — by symmetry we can assume without loss of generality $[c, d] = R$.

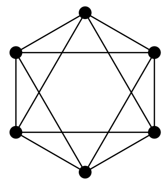
Hence $[d, e] = G$, and so $[b, d] = Y$ and then $[b, c] = B$, and now a fifth colour is needed for $[c, e]$. Hence $\chi'(G) = 5$.

16. Find the chromatic index of the cube, and of the octahedron.

Solution.

The graphs are as shown:





By Vizing's theorem, the cube graph has chromatic index 3 or 4 (3 being the maximum vertex degree). Similarly the octahedral graph has chromatic index 4 or 5. In fact the *lower* value is correct in each case, as the following edge colourings (thick, thin, dotted, dashed) show: