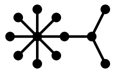
The University of Sydney
MATH2969/2069

Graph Theory **Tutorial 6 (Week 13) Solutions** 2008

1. Find $\chi(G)$ and $\chi'(G)$ in each of the following cases.

(i) $G = K_9$ (ii) $G = K_{10}$

(iii) $G = K_{5,6}$ (iv) $G =$ 

Solution.

(i) For a proper (vertex) colouring of a complete graph, each vertex clearly must be assigned a different colour. So $\chi(K_9) = 9$.

The chromatic index of K_n is n if n is odd, and $n-1$ if n is even. So $\chi'(K_9) = 9$.

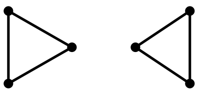
(ii) $\chi(K_{10}) = 10$; $\chi'(K_{10}) = 9$.

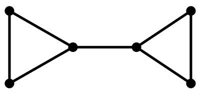
(iii) By definition, the chromatic number of a bipartite graph is two. The chromatic index of a bipartite graph is the maximum vertex degree.

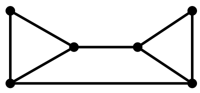
So $\chi(K_{5,6}) = 2$ and $\chi'(K_{5,6}) = 6$.

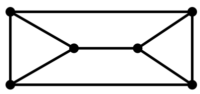
(iv) A tree is a bipartite graph. Therefore $\chi(G) = 2$ and $\chi'(G) = 8$.

2. Which of the following graphs are orientable?


(a)


(b)


(c)


(d)

Solution.

The graph in (a) is not connected, and therefore not orientable. The graph in (b) is not orientable, since it has a bridge. The graphs in (c) and (d) are connected, with no bridges, and are therefore orientable.

3. Are either of the following statements true?

(i) Any Hamiltonian graph is orientable.

(ii) Any orientable graph is Hamiltonian.

Solution.

(i) This statement is true. If a graph is Hamiltonian, and the Hamiltonian cycle is oriented consistently in one direction, then there is a directed path from any vertex to any other vertex around the Hamiltonian cycle. (Alternately, a Hamiltonian graph must be connected, and have no bridges.)

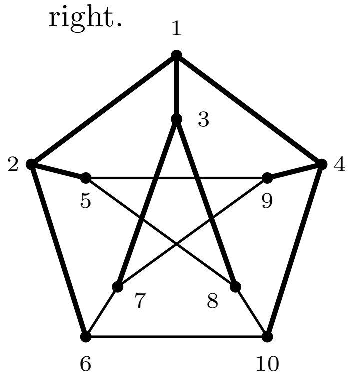
(ii) This statement is not true. Many graphs are orientable (that is, connected and without bridges) but not Hamiltonian. The graph $K_{2,3}$ is a simple example.

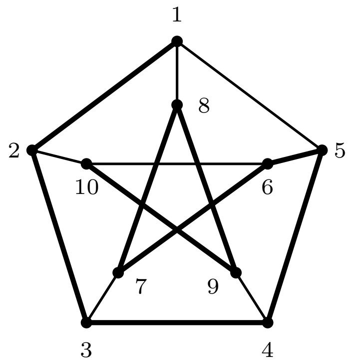
4. Perform a breadth first search, and a depth first search on the Petersen graph. Find a strongly connected orientation for the Petersen graph.

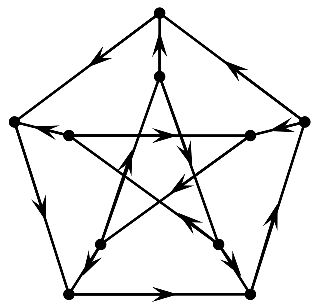
Solution.

The spanning trees which result from the searches are indicated by thick lines in the following diagrams, with the breadth first search shown on the graph on the left, and the depth first search shown on the graph in the middle. The labels indicate the order in which vertices were chosen. (There are, of course, other solutions.)

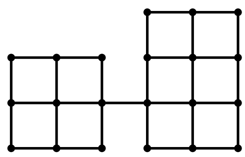
A strongly connected orientation can be found using the depth-first search tree. Orient each edge in the tree from smaller label to larger label, and each edge not in the tree from larger label to smaller label, as shown in the graph on the right.







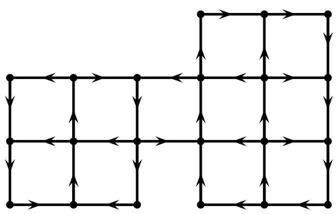
5. Explain why the addition of exactly one edge will make this graph orientable. Add the edge, and then find a strongly connected orientation.



Solution.

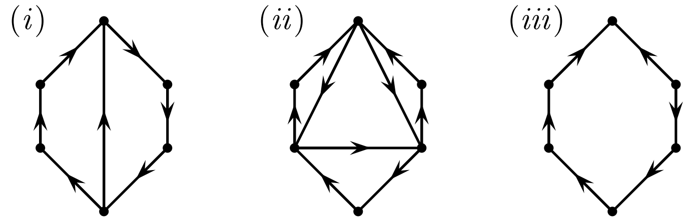
The graph is connected, but has a bridge, and therefore is not orientable. By adding exactly one edge (in an appropriate place, of course), the graph will no longer have a bridge and will therefore be orientable. There are, obviously, lots of pairs of vertices between which an edge could be added so that the rsulting graph is orientable. The diagram below gives one example. (Possibly the most obvious thing to do is add another edge between the vertices at either end of the bridge – equivalent to making the bridge two-way if we're thinking in terms of street maps.)

Once the edge has been added, a strongly connected orientation can be found using a depth-first search tree, as in question 3. The diagram shows one possible orientation.



6. Determine which of the following graphs are

(a) strongly connected; (b) Eulerian; (c) Hamiltonian.



Solution.

(*i*) The graph is strongly connected, since any vertex can be reached from any other by following the directed cycle around the hexagon. Furthermore, this cycle visits each vertex in the graph, and therefore the graph is Hamiltonian.

There are two vertices such that the in-degree does not equal the out-degree, and so the graph is not Eulerian.

(*ii*) Every edge in the graph is part of a cycle, and so the graph is srongly connected.

The in-degree equals the out-degree for every vertex, and so the graph is Eulerian.

The graph is not Hamiltonian. A Hamiltonian cycle would have to use all six of the edges incident at the vertices with in-degree and out-degree both equal to 1. But then it would have to go into, and out of, the vertex at the top of the graph twice.

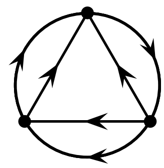
(*iii*) The graph is clearly neither strongly connected, nor Eulerian, nor Hamiltonian.

7. (*i*) Is this digraph simple?

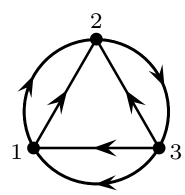
(*ii*) Is it Eulerian?

(*iii*) Is it Hamiltonian?

(*iv*) Is it strongly connected?



Solution.



(*i*) The digraph is not simple. There are two arcs from vertex 1 to vertex 2, for example.

(*ii*) The out-degree of vertex 2 is 1, and the in-degree is 3. Therefore the graph is not Eulerian.

(*iii*) There is a Hamiltonian cycle from vertex 1 to 2 to 3 to 1.

(*iv*) The graph is strongly connected. There is a directed path from each vertex to every other vertex around the Hamiltonian cycle, for example.

8. In a tournament, the *score* of a vertex is its out-degree, and the *score sequence* is a list of all the scores in non-decreasing order.

(i) If the score sequence of a tournament is $(s_1, s_2, \ldots, s_n)$, show that $\sum_{i=1}^{n} s_i = \frac{n(n-1)}{2}$.

(ii) What is the score sequence of a tournament with n vertices if each vertex has a different score?

(iii) Find a semi-Hamiltonian path in a tournament with n vertices $v_0, v_1, \ldots, v_{n-1}$ if the score of vertex v_i is i.

Solution.

(i) In any digraph the sum of the out-degrees is equal to the sum of the in-degrees. (Every edge has to go out of one vertex and into another.) The sum of all the out-degrees and all the in-degrees is, of course, twice the number of edges (by the hand-shaking lemma). Therefore, the sum of the out-degrees is equal to the number of edges.

Now, if the score sequence of a tournament is $(s_1, s_2, \ldots, s_n)$, then the tournament has n vertices, and

$$\sum_{i=1}^{n} s_i = \text{the sum of the out-degrees} = \text{the number of edges.}$$

But a tournament with n vertices has $\frac{n(n-1)}{2}$ edges, since each vertex is joined to every other vertex by exactly one directed edge. The result therefore follows.

(ii) The score, or out-degree, of any vertex in a tournament with n vertices is at most $n-1$. Clearly, the score of a vertex is not negative, so any score s is such that $0 \leq s \leq n-1$. The only set of n different integers in this range is $\{0, 1, 2, \ldots, n-1\}$. So the score sequence is $(0, 1, 2, \ldots, n-1)$.

(iii) Consider vertex v_{n-1}. Its out-degree is $n-1$, and so there is an edge directed from v_{n-1} to every other vertex, and in particular to v_{n-2}. Vertex v_{n-2} has out-degree $n-2$, and in-degree 1 (from v_{n-1}). There is therefore an edge directed from v_{n-2} to each of $v_0, v_1, \ldots, v_{n-3}$. In particular, there is an edge directed from v_{n-2} to v_{n-3}. Continuing in this way, we see that for each i, $1 \leq i \leq n-1$, there is an edge directed from v_i to v_{i-1}. Therefore $v_{n-1}v_{n-2}\ldots v_0$ is a path containing all the vertices – that is, a semi-Hamiltonian path.