SAS 11/11/2020

Monday, 9 November 2020 1:02 PM

The Chern Character

Reminder:

The Chern classes are the unique sequence of functions $c_1, c_2, \cdots$ assigning to each vector bundle $E \longrightarrow X$ a class $c_i(E) \in H^{2i}(X, \mathbb{Z})$ such that:

1) $c_i(f^*E) = f^* c_i(E)$
2) $c(E \oplus F) = c(E) \smile c(F)$ for $c = 1 + c_1 + c_2 + \cdots$
3) $c_i(E) = 0$ for $i > \dim(E)$
4) For the tautological line bundle $E_1(\mathbb{C}P^1) \longrightarrow \mathbb{C}P^1$, $c_1(E_1(\mathbb{C}P^1))$ is a generator of $H^2(\mathbb{C}P^1, \mathbb{Z}) = \mathbb{Z}$

c_1 has the property (Hatcher Prop 3.10)

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

Aim: Define a ring homomorphism

$$ch : K(X) \longrightarrow H^{even}(X, \mathbb{Q}).$$

Let $L \to X$ be a line bundle and define

$$ch(L) = \exp(c_1(L))$$
$$= 1 + c_1(L) + c_1(L)^2/2! + \cdots \in H^{even}(X, \mathbb{Q})$$

This is a good choice because our ring structure on $K(X)$ is given by $\otimes$ and $\oplus$ so using this defn gives

gives

$$ch(L\otimes L') = \exp(c_1(L\otimes L'))$$
$$= \exp(c_1(L)+c_1(L'))$$
$$= \exp(c_1(L))\cdot\exp(c_1(L'))$$
$$= ch(L)\cdot ch(L')$$

Then if a vector bundle $E\to X$ decomposes as $E = L_1\oplus\cdots\oplus L_k$ we can define

$$ch(E) = ch(L_1)+\cdots+ch(L_k)$$
$$= \sum_i \frac{1}{i!}(c_1(L_1)^i+\cdots+c_1(L_k)^i)$$

Using the property of $c(E\oplus F) = c(E)\smile c(F)$, the total Chern class of E is

$c(E) = (1+x_1)\cdots(1+x_k)$ where $x_i = c_1(L_i)$

$$= 1 + \underbrace{(x_1+\cdots+x_k)}_{=\sigma_1} + \underbrace{(x_1x_2+x_1x_3+\cdots+)}_{=\sigma_2} + \underbrace{(x_1x_2x_3+\cdots)}_{=\sigma_3} + \cdots$$
$$= 1 + c_1(E) + c_2(E) + c_3(E) + \cdots$$

$$c_i(E) = \sigma_i(x_1,\cdots,x_k)$$

Hatcher Section 2.3 we have the Newton polynomials which satisfy

$$x_1^i+\cdots+x_k^i = s_i(\sigma_1(x_1,\cdots,x_k),\cdots,\sigma_i(x_1,\cdots,x_k))$$
$$= s_i(c_1(E),\cdots,c_i(E))$$

We can also write

$$\boxed{ch(E) := \sum_i \frac{s_i(c_1,\cdots,c_i)}{i!}}$$

This formula does not require E to decompose into line bundles.

Proposition: $ch(E \oplus F) = ch(E) + ch(F)$ and
$ch(E \otimes F) = ch(E) \cdot ch(F)$

In general, vector bundles do not split into line bundles. However, we can pull back to one that does.

Theorem (Splitting principle): Let $E \to X$ be a vector bundle. Then there exists a space $Fl(E)$ and a map $\phi : Fl(E) \longrightarrow X$ such that
a) $\phi^* E = L_1 \oplus \cdots \oplus L_k$ and
b) $\phi^* : H^\bullet(X, \mathbb{Q}) \longrightarrow H^\bullet(Fl(E), \mathbb{Q})$ is injective.

Proof of proposition: Suppose that $E = \bigoplus_{i=1}^{k} L_i$ and $F = \bigoplus_{i=1}^{\ell} \hat{L}_i$. Then

$$\begin{aligned}
ch(E \oplus F) &= ch(L_1 \oplus \cdots \oplus L_k \oplus \hat{L}_1 \oplus \cdots \oplus \hat{L}_\ell) \\
&= ch(L_1) + \cdots + ch(L_k) + ch(\hat{L}_1) + \cdots + ch(\hat{L}_\ell) \\
&= ch(E) + ch(F) \\
ch(E \otimes F) &= ch\left(\left[\bigoplus_{i=1}^{k} L_i\right] \otimes \left[\bigoplus_{i=1}^{\ell} \hat{L}_i\right]\right) \\
&= ch\left(\bigoplus_{i,j} (L_i \otimes \hat{L}_j)\right) \\
&= \sum_{i,j} ch(L_i \otimes \hat{L}_j) \\
&= \sum_{i,j} ch(L_i) \cdot ch(\hat{L}_j) \\
&= \left(\sum_i ch(L_i)\right) \cdot \left(\sum_j ch(\hat{L}_j)\right) \\
&= ch(E) \cdot ch(F)
\end{aligned}$$

Now consider general vector bundles E, F.

$$\begin{tikzcd}
E \oplus F \arrow[r] & X \\
L_1 \oplus \cdots \oplus L_k \oplus \phi_E^* F \arrow[r] & Fl(E) \arrow[u, "\phi_E"']
\end{tikzcd}$$

$$\begin{tikzcd} & \\ L_1'\oplus\cdots\oplus L_k' \oplus \hat{L}_1 \oplus\cdots\oplus \hat{L}_l \arrow[r] & Fl(\phi_E^* F) \arrow[u, "\phi_{\phi_E^* F}"'] \end{tikzcd}$$

So we can use the map $\phi = \phi_E \circ \phi_{\phi_E^* F}$ to split $E\oplus F$, E and F simultaneously. Now apply ϕ^*

$$\begin{aligned} \phi^*(ch(E\oplus F)) &= ch(\phi^* E \oplus \phi^* F) \\ &= ch(\phi^* E) + ch(\phi^* F) \\ &= \phi^*(ch(E) + ch(F)) \end{aligned}$$

which implies

$$ch(E\oplus F) = ch(E) + ch(F)$$

by the injectivity of $\phi^* : H^\bullet(X) \to H^\bullet(Fl(\phi_E^* F))$

Similarly for $\otimes$.

$\blacksquare$

Then define $ch: K(X) \to H^{even}(X, \mathbb{Q})$ by

$$ch(E-F) = ch(E) - ch(F)$$

which is a homomorphism by the previous prop^n:

$$\begin{aligned} ch((E-F)+(\hat{E}-\hat{F})) &= ch((E\oplus\hat{E})-(F\oplus\hat{F})) \\ &= ch(E) + ch(\hat{E}) - ch(F) - ch(\hat{F}) \\ &= ch(E-F) + ch(\hat{E}-\hat{F}) \end{aligned}$$

$$\begin{aligned} ch((E-F)(\hat{E}-\hat{F})) &= ch(E\otimes\hat{E}) - ch(E\otimes\hat{F}) \\ &\quad - ch(F\otimes\hat{E}) + ch(F\otimes\hat{F}) \\ &= ch(E-F)\cdot ch(\hat{E}-\hat{F}) \end{aligned}$$

Theorem: We have the following

a) if $X = S^{2n}$, then ch is injective with image equal to $H^\bullet(S^{2n}, \mathbb{Z}) \subseteq H^\bullet(S^{2n}, \mathbb{Q})$.

b) If X is a cell complex, then ch induces an isomorphism

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$$K(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^{even}(X, \mathbb{Q})$$

Theorem: TS^{2n} does not admit the structure of a complex vector bundle for $n \neq 1, 2$ or 3

proof:

Let $E \longrightarrow S^{2n}$ be a complex vector bundle. Then $ch(E) \in H^\bullet(S^{2n}, \mathbb{Z}) \subset H^\bullet(S^{2n}, \mathbb{Q})$. Since $c_i(S^{2n}) = 0$ for $i = 1, \cdots, n-1$ we have

$$\begin{aligned} ch(E) &= \sum_i s_i(c_1, \cdots, c_i) \\ &= s_n(c_1, \cdots, c_n) \\ &= (-1)^{n-1} c_n(E)/(n-1)! \\ &= (-1)^{n-1} e(E)/(n-1)! \end{aligned}$$

If $E = TS^{2n}$, then $e(E) = \chi(S^{2n})$, the Euler characteristic of S^{2n}. Even dimensional spheres have $\chi(S^2) = 2$, thus $(n-1)!$ divides 2. Thus $n = 1, 2$ or 3.

$\square$

Corollary: S^{2n} does not admit the structure of a complex manifold for $n \neq 1$

What about $n = 1, 2$ or 3

$n=1$, $S^2 = \mathbb{C}P^1$ which is a 1-dimensional complex manifold so $T\mathbb{C}P^1$ is a complex vector bundle.

$n=2$, TS^4 is not a complex vector bundle. See Karoubi.

$n=3$, Open problem. See "Hopf problem".

References:

- Bott R., Tu L.W. - "Differential forms in Algebraic Topology." page 277
  - Some useful material on the Splitting principle
- Hatcher A. - "Vector Bundles and K-Theory"
- Karoubi M. - "K-Theory - An Introduction" page 310
  - Result on TS^{2n}.
- Lawson H., Michelsohn M. - "Spin Geometry"
  - Further discussion on the Chern Character.