Recall: We've talked about fiber bundles with fiber F.

Let $p: E \longrightarrow B$ be a vector bundle of rank n / $\mathbb{C}$.

Example

① projective bundle $P(E) \longrightarrow B$

$P(E)$: lines through the origin in fibers of E

$\longrightarrow$ $P(E)$ is a fiber bundle with fiber $\mathbb{C}P^{n-1}$.

② flag bundle $F(E) \longrightarrow B$

$F(E) \subseteq P(E)^{\times n}$: n-tuples of orthogonal lines through the origin of fibers of E

$\longrightarrow$ $F(E)$ has fiber flag manifold $F(\mathbb{C}^n)$

vector bundle $\quad E \longrightarrow B$

$\downarrow$

flag bundle $\quad F(E) \longrightarrow B$

$\quad \rotatebox{90}{$\in$}$

$(b, \ell_1, \ell_2, \dots, \ell_n \subseteq E_b)$

$\downarrow$

i^{th} line : line bundle $\quad L_i \longrightarrow F(E)$

$\quad \rotatebox{90}{$\in$}$

$(b, \ell_1, \ell_2, \dots, \ell_n, v \in \ell_i)$

$$(b, \ell_1, \dots, \ell_n, v_i \in \ell_i \ \forall i) \Leftrightarrow (b, \overbrace{\ell_1, \ell_2, \dots, \ell_n}^{\text{span } E_b}, v \in E_b)$$

$$\begin{tikzcd} L_1 \oplus \cdots \oplus L_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ F(E) \arrow[r] & B \end{tikzcd}$$

pullback

Conclusion: Can **always** pull back an arbitrary bundle to a sum of line bundles.

All top. spaces in this talk are assumed compact and Hausdorff.

The splitting principle

Given a vector bundle $E \longrightarrow X$,

$\exists$ map $p: Y \longrightarrow X$ such that

- $p^*(E)$ is a sum of line bundles
- $K(X) \xrightarrow{p^*} K(Y)$ is injective

Corollaries we've used

- existence of Adams operations
- Adams' theorem

The proof will use the

- Leray-Hirsch theorem
- computation of $K^*(\mathbb{C}P^n)$

Let's "compute" $K^*(\mathbb{CP}^n)$

Proposition

① If X is a finite cell complex with n cells, $K^*(X)$ is a finitely generated group with $\leq n$ generators.

② If all the cells have even dimension, then $K^1(X) = 0$ and $K^0(X)$ is free abelian with 1 basis element for each cell.

PROOF

by induction on # cells.

Suppose X is built from Y by attaching a k-cell. $Y \hookrightarrow X \rightarrow X/Y = S^k$

① $$\underset{\underset{\mathbb{Z}}{\parallel}}{\tilde{K}^*(X/Y)} \rightarrow \tilde{K}^*(X) \rightarrow \tilde{K}^*(Y)$$

Let's "compute" $K^*(\mathbb{CP}^n)$

Proposition

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PROOF

②

$$\begin{tikzcd}
 & K^1(X/Y) \arrow[r] \arrow[d, equal] & K^1(X) \arrow[r] \arrow[d, equal] & K^1(Y) \arrow[d, equal] \arrow[dddll, bend left] & \\
 & K^1(S^k) \arrow[d, equal] & 0 & 0 & \\
 & 0 & & & \\
\text{Hence} & K^0(X/Y) \arrow[r, hook] \arrow[d, equal] & K^0(X) \arrow[r] & K^0(Y) \arrow[r] & 0 \\
 & \mathbb{Z} & & &
\end{tikzcd}$$

SES splits since $K^0(Y)$ is free. $\square$

Let's "compute" $K^*(\mathbb{C}P^n)$

$\mathbb{C}P^n$ has cell structure with one cell in each dimension $0, 2, 4, \dots, 2n$

so $$\left.\begin{aligned} K^1(\mathbb{C}P^n) &= 0 \\ K^0(\mathbb{C}P^n) &= \mathbb{Z}^{n+1} \end{aligned}\right\}$$ by proposition

Proposition $K(\mathbb{C}P^n) = \mathbb{Z}[L] / (L-1)^{n+1}$

where L is the canonical line bundle over $\mathbb{C}P^n$.

Sketch of proof

By induction on n.

Base step: $K(\mathbb{C}P^1) = K(S^2) = \mathbb{Z}[L]/(L-1)^2$ ✓

$\mathbb{C}P^{n-1} \subseteq \mathbb{C}P^n$

attach 2n-cell

We have SES

$$0 \to K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \to K(\mathbb{C}P^n) \longrightarrow K(\mathbb{C}P^{n-1}) \to 0$$

$K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) = \tilde{K}(S^{2n})$, $\quad K(\mathbb{C}P^{n-1}) = \mathbb{C}[L]/(L-1)^n$

Want to show: $(L-1)$ generates $K(\mathbb{C}P^n, \mathbb{C}P^{n-1})$.

By Bott periodicity:

$$\tilde{K}(S^2) \otimes \dots \otimes \tilde{K}(S^2) \longrightarrow \tilde{K}(S^{2n})$$

$\tilde{K}(S^{2n})$ is $K(\mathbb{C}P^n, \mathbb{C}P^{n-1})$

$$(L-1) \otimes \dots \otimes (L-1) \longmapsto (L-1)^n$$

Leray-Hirsh theorem

Let $p : \overset{F}{\underset{\cap}{Z}} \longrightarrow B$ be a fiber bundle with fiber F. Suppose

- $K^*(F)$ is free
- $\exists\, c_1, \dots, c_k \in K^*(Z)$ giving a basis for $K^*(F)$ under $K^*(Z) \to K^*(F)$ in each fiber
- F is a finite cell complex with all cells of even dimension (e.g. $\mathbb{C}P^n$)

Then $K^*(Z)$ is free as a module over $K^*(B)$, with basis $c_1, \dots, c_k$.

$K^*(B) \xrightarrow{p^*} K^*(Z)$

PROOF See **Hatcher** section 2.3.

Example

vector bundle $\qquad E \longrightarrow X$

$\downarrow$

projective bundle $\qquad P(E) \xrightarrow{p} X$

with fiber $F = \mathbb{C}P^{n-1}$ $\qquad \ni (x, \ell \subseteq E_x)$

$\downarrow$

line bundle $\qquad L \longrightarrow P(E)$

$\ni (x, \ell \subseteq E_x, v \in \ell)$

$K^*(E_x)$

$$K^*(P(E)) \longrightarrow K^*(\mathbb{C}P^{n-1}) = \mathbb{Z}[L]/(L-1)^n$$

$$L \longmapsto \text{canonical line bundle } L$$

$$\{1, L, L^2, \dots, L^{n-1}\} \longmapsto \text{basis of } K^*(\mathbb{C}P^{n-1})$$

$\Rightarrow$ LERAY HIRSCH: $K^*(P(E))$ is FREE over $K^*(X)$ with basis $\{1, L, L^2, \dots, L^{n-1}\}$

The splitting principle

Given a vector bundle $E \longrightarrow X$,

$\exists$ map $p : Y \longrightarrow X$ such that

- $p^*(E)$ is a sum of line bundles
- $K^*(X) \xrightarrow{p^*} K^*(Y)$ injective

PROOF

goal: find p to split of one line bundle

$p^*(E) = L \oplus E'$ (L line bundle)

then repeat procedure for E' etc...

Consider $p^* : K^*(X) \longrightarrow \underbrace{K^*(P(E))}$

free over $K^*(X)$ with basis $1, L, \dots, L^{n-1}$

Clearly p^* is injective.

$$p^* : K^*(X) \longrightarrow K^*(P(E))$$

$$E \longmapsto p^*(E)$$

Note that

$$(x, \ell \subseteq E_x, v \in E_x)$$

$$\begin{tikzcd}
 & p^*(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\
(x, \ell \subseteq E_x, v \in \ell) \in L \arrow[ur, hook] \arrow[r] & P(E) \arrow[r] & X
\end{tikzcd}$$

Hence $p^*(E) = L \oplus E'$ for

$$\begin{tikzcd}
E' \arrow[rr, hook] \arrow[dr] & & p^*(E) \arrow[dl] \\
 & P(E) &
\end{tikzcd}$$

(E' is the subbundle of $p^*(E)$ orthogonal to L for some choice of inner product)

$\Rightarrow$ now repeat the process for $E' \to P(E)$

that is find $P(E')$ etc...

$\Rightarrow$ pairs of orthogonal lines through the origin in fibers of E

After finitely many steps:

we find $Y = F(E)$ flag bundle,

and

- $p^*(E)$ is a sum of line bundles
- $K^*(X) \xrightarrow{p^*} K^*(Y)$ injective.

$\square$