## The plan for Today

1. The functors $K$, $\tilde{K}$

2. The fundamental product Theorem.

## Assumptions for today:

1. $\mathbb{k} = \mathbb{C}$ (not $\mathbb{R}$!)

2. $X$ is compact Hausdorff.

3. $\dim p^{-1}(x)$ needn't be globally constant.

# The functors $K$, $\tilde{K}$.

## Direct Sums Revisited

Recall:

$$\begin{tikzcd} E_1 \oplus E_2 := E_1 \times_X E_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \arrow[d] \\ E_2 \arrow[r] & X \end{tikzcd}$$

Base

V. Bun's over X.

As seen in Josh's talk:

$$NS^n \oplus TS^n \cong \varepsilon_{n+1}$$

Normal Bundle to n-sphere

Tangent Bundle to n-sphere

rank $n+1$ Trivial bundle of n-sphere.

More generally the following is true:

**Proposition:** $\forall\ E \to B$ (compact, Hausdorff),

$\exists\, E' \to B$, s.t. $E \oplus E' \cong \varepsilon_N$

<u>Proof:</u> $\forall x \in B$, $\exists\, U_x \overset{\text{open}}{\subset} B$ s.t. $p^{-1}(U_x)$

is locally trivial.

Hausdorff + compact $\Rightarrow$

$\exists \{ \varphi_x : U_x \to [0,1] \mid x \in \text{support}(\varphi_x) \subset U_x \}$.

$\Rightarrow \{\text{support}(\varphi_x)\}$ covers $B$.

$\overset{\text{compact}}{\Rightarrow}$ $\exists \{ U_i := U_{x_i} \mid i \in \text{finite set} \}$ covers $B$.

Then define $g_i : E \to \mathbb{C}^n$

$$g_i(v) = \varphi_i(p(v)) \cdot (\pi h_i(v)),$$

where $E \xrightarrow{p} B$, $p^{-1}(U_i) \xrightarrow[\text{local Trivialisation.}]{h_i} U_i \times \mathbb{C}^n$

$U_i \times \mathbb{R}^n \xrightarrow[\text{projection.}]{\pi} \mathbb{C}^n$

Then $g_i|_{p^{-1}(\text{support}(\varphi_i))}$ is a linear injection on each fibre.

Take $g_i$ to be the coordinate functions of

$g: E \longrightarrow \mathbb{C}^N$ ← n × cardinality of indexing set of $\{U_i\}$ is a linear injection on fibres.

Define $f: E \xrightarrow{p \times g} B \times \mathbb{C}^N$

Then $E \cong im(f) \underset{\text{bundle}}{\overset{\text{sub}}{\subset}} B \times \mathbb{C}^N$

$\therefore \quad \exists E^\perp \underset{\text{bundle.}}{\overset{\text{sub}}{\subset}} B \times \mathbb{C}^N$ s.t. $E \oplus E^\perp \cong B \times \mathbb{C}^N$.

(exercise).

$\blacksquare$

## Equivalence Relations:

Given $E_1 \xrightarrow{p_1} X \xleftarrow{p_2} E_2$ define:

- $E_1 \simeq_s E_2$ if $E_1 \oplus \varepsilon_n \cong E_2 \oplus \varepsilon_n$

"stably isomorphic" for some $n \in \mathbb{Z}_{\geq 0}$.

- $E_1 \sim E_2$ if $E_1 \oplus \varepsilon_n \cong E_2 \oplus \varepsilon_m$

for some $n, m \in \mathbb{Z}_{\geq 0}$.

These form equivalence classes of vector bundles on $X$.

## Group Structure:

**Proposition:** $\sim$-equivalence classes of vector bundles on $X$ (Compact, Hausdorff) form an abelian group called $\tilde{K}(X)$.
"Reduced K-theory".

**Proof:** Easy to show:

- $\oplus$ is well defined, commutative and associative.
- $\varepsilon_0$ is the zero element.

Existence of inverses is the previous proposition.

(note: Can modify proof to include non-constant dimensions of fibres). ▨

Can't naively endow $\mathrm{Vect}(X)/_{\simeq_s}$ with a group structure since inverses usually don't exist.

$$E_1 \oplus E_2 \simeq_s \varepsilon_0$$

$$\Rightarrow E_1 \oplus E_2 \oplus \varepsilon_n \simeq \varepsilon_n \quad \text{for some } n \in \mathbb{Z}_{\geq 0}.$$

$\Leftrightarrow$ $E_1$, $E_2$ have dimension $0$.

Does satisfy the cancellation property. (Requires Compactness)

Proposition: /Definition: Equivalence classes of formal differences of vector bundles on $X$ form an abelian group $K(X)$, where

$E_1 - F_1$ is equivalent to $E_2 - F_2$ if

$E_1 \oplus F_2 \cong_s E_2 \oplus F_1$. Where the

group operation is:

$(E_1 - F_1) + (E_2 - F_2) = (E_1 \oplus E_2) - (F_1 \oplus F_2)$.

Proof: (Exercise) Show:

- $\forall E \in Vect(X)$, $E - E$ is the equivalence class of the zero element.
- $\forall E, F \in Vect(X)$, the inverse of $E - F$ is $F - E$.
- $\oplus$ is well defined etc. $\blacksquare$

Proposition: We have a split short exact sequence

$$0 \longrightarrow K(x_0) \hookrightarrow K(X) \twoheadrightarrow \tilde{K}(X) \twoheadrightarrow 0$$

So $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$. Non Canonically.

Proof: First define

$$K(X) \longrightarrow \tilde{K}(X)$$

$$E - \varepsilon_n \longmapsto [E]_\sim$$

Well defined since

$$E - \varepsilon_n = F - \varepsilon_m \in K(X)$$

$$\Rightarrow E \oplus \varepsilon_m \cong_s F \oplus \varepsilon_n \in Vect(X)$$

$$\Rightarrow E \sim F$$

$$\Rightarrow [E]_\sim = [F]_\sim \in \tilde{K}(X).$$

Observe $K(X) \to \tilde{K}(X)$ is surjective.

Kernel consists of:

$$E - \varepsilon^n \quad \text{s.t.} \quad E \sim \varepsilon_0$$

$\Rightarrow E \simeq_s \mathcal{E}_m$ for some $m \in \mathbb{Z}_{\geq 0}$

$\Rightarrow \text{Ker} = \{ \mathcal{E}_n - \mathcal{E}_m \mid n, m \in \mathbb{Z}_{\geq 0} \}$

$\cong \mathbb{Z}.$

To find a splitting we must choose a basepoint $x_0 \in X$. Restriction gives

$$K(X) \twoheadrightarrow K(x_0) \cong \mathbb{Z}$$

This is an iso on $\{\mathcal{E}_n - \mathcal{E}_m\}$

$$\therefore \quad K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}.$$

<u>R(i)ng Structures:</u>

Obvious thought: use $\otimes$ as multiplication.

<u>Proposition:</u> $K(X)$ has a commutative ring structure where multiplication is given by:

$$(E_1 - F_1)(E_2 - F_2) = E_1 \otimes E_2 - E_1 \otimes F_2 - F_1 \otimes E_2 + F_1 \otimes F_2$$

**Proof:** (Exercise)

- check well defined.
- Commutative
- $\mathcal{E}_1$ is identity.

**Proposition:** $\tilde{K}(X)$ has a commutative rng structure. (Non canonically?).

needn't have a unit.

**Proof:** Restriction gives a hom $K(X) \to K(x_0)$ with kernel $\tilde{K}(X)$. Being an ideal of $K(X)$ induces the rng structure.

**Functorial Properties:**

**Proposition:** $f : X \to Y$ induces a ring hom:

Cts map

$$f^* : K(Y) \to K(X)$$
$$E - F \mapsto f^*(E) - f^*(F).$$

<u>Proof:</u> Follows from properties of pull back:

$$f^*(E \oplus F) \cong f^*(E) \oplus f^*(F)$$
$$f^*(E \otimes F) \cong f^*(E) \otimes f^*(F).$$

$\blacksquare$

$f^*$ also satisfies functorial properties:

$$(f\,g)^* = g^* f^*$$

$$\mathbb{1}^* = \mathbb{1}$$

$$f \simeq g \Rightarrow f^* = g^*$$

Homotopic maps.

map of pointed spaces

Prop: $f : (X, x_0) \longrightarrow (Y, y_0)$ induces a ring hom $f^* : \tilde{K}(Y) \longrightarrow K(X)$

Proof: as above.

## External Product & Fundamental Product thm:

**Def** We have an *external product*

$$\mu : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$

$$a \otimes b \longmapsto p_X^*(a)\, p_Y^*(b)$$

projection maps from $X \times Y$ onto $X$ (resp. $Y$).

$K(X) \otimes K(Y)$ is a ring with group-like multiplication: $(a \otimes b)(c \otimes d) = ac \otimes bd$.

$\therefore$ $\mu$ is a ring hom.

**Theorem:** For any $X$, compact, Hausdorff

$$\mu' : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \xrightarrow{\sim} K(X \times S^2).$$

is an isomorphism of rings.