K-THEORY: LECTURE 3: Bott Periodicity

Giulian Wiggins

Last time:

- If B is a compact space, then $K(B)$ is the additive group with
  - generators: $[E]$ for each complex vector bundle $E \to B$
  - relations: $[E]+[E'] = [E \oplus E']$
- Recall: In $K(B)$ a typical element is of the form $[E_1]-[E_2]$.

  Moreover: $[E_1]-[E_2] = [E_3]-[E_4]$ iff for some $E_5 \to B$,

$$E_1 \oplus E_4 \oplus E_5 = E_2 \oplus E_3 \oplus E_5 .$$

(Actually: may assume E_2, E_4, E_5 are trivial bundles in the above statement - see first prop. in Joe's talk)

- Examples: $K(pt) = \mathbb{Z}$, $K(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$, $K(S^{2n+1}) = \mathbb{Z}$

- Recall: For $x \in B$ the maps $\{x\} \overset{i}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} \{x\}$ give maps

$$\begin{tikzcd} \mathbb{Z} = K(\{x\}) \arrow[r, "p^*"] \arrow[dr, "id"'] & K(B) \arrow[d, "i^*"] \\ & K(\{x\}) = \mathbb{Z} \end{tikzcd}$$

So $K(B)$ splits as a direct sum:

$$K(B) = \tilde{K}(B) \oplus \mathbb{Z}$$

(here $\tilde{K}(B) = \ker(K(B) \to \mathbb{Z})$)

- Recall: tensor product of vector bundles $\rightsquigarrow$ multiplication on $K(B)$

Aim: Want to think of $\tilde{K}(B)$ as the zeroth part of a (reduced) generalized cohomology theory.

We will define a cohomology theory properly later, but so that we have some idea where we are heading, here is the definition of a (relative) cohomology theory for CW-pairs:

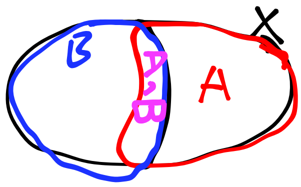
Data: A sequence of contravariant functors
$$h^i : \text{CW-pairs}^{op} \longrightarrow \mathbb{Z}\text{-Mod}$$
and a natural transformation $d : h^i(A) := h^i(A,\emptyset) \to h^{i+1}(X,A)$ (called the boundary morphism)

Axioms:

① Homotopy: Each h^i takes homotopic maps to homotopic maps.

② Exactness: Each pair (X,A) induces a long exact sequence
$$\cdots \to h^i(X,A) \xrightarrow{f^*} h^i(X) \xrightarrow{g^*} h^i(A) \xrightarrow{d} h^{i+1}(X,A) \to \cdots$$
where $(A,\emptyset) \xhookrightarrow{g} (X,\emptyset) \xhookleftarrow{f} (X,A)$ $\quad (f^* = h^i(f))$

③ Excision: If $X = A \cup B$ then the inclusion $(A, A\cap B) \xhookrightarrow{f} (X,B)$ induces an isomorphism $h^i(X,B) \xrightarrow[\sim]{f^*} h^i(A, A\cap B)$



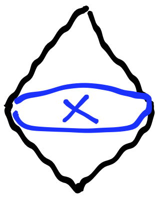
④ Additivity: If $(X,A) = \coprod_\alpha (X_\alpha, A_\alpha)$ then $h^i(X,A) \simeq \prod_\alpha h^i(X_\alpha, A_\alpha)$

Background from algebraic topology

Given a topological space X, define the <u>suspension</u> of X, SX, to be the quotient space:

$$SX = X \times I/\sim \qquad (I = [0,1])$$

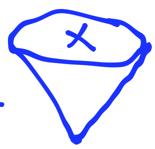
where $(x_1,0) \sim (x_2,0)$ and $(x_1,1) \sim (x_2,1)$ for all $x_1, x_2 \in X$.

i.e. $SX =$ 

e.g. $S(\underbrace{S^m}_{m\text{-sphere}}) = S^{m+1}$.

<u>Prop.</u> (i) $\tilde{H}^n(X) = \tilde{H}^{n+1}(SX)$

(ii) $\tilde{H}_n(X) = \tilde{H}_{n+1}(SX)$

<u>proof</u>: Let $CX =$  be the cone on X. Then by 'exactness' we have a long exact sequence:

$$\cdots \to \tilde{H}^n(CX) \to \tilde{H}^n(X) \to \tilde{H}^{n+1}(CX,X) \to \tilde{H}^{n+1}(CX) \to \cdots$$

By excision: $\tilde{H}(CX,X) = \tilde{H}(SX)$.

Since CX is contractible: $\tilde{H}^n(CX) = 0 \ \forall n$

So $\tilde{H}^n(X) = \tilde{H}^{n+1}(SX)$.

<u>NB.</u> This proof only relies on axioms for a generalised cohomology theory. So it follows that in any generalised cohomology theory $\{h^i\}_{i\in\mathbb{Z}}$ we have isomorphisms:

$$\tilde{h}^i(X) = \tilde{h}^{i+1}(SX).$$

This motivates the definition of $\tilde{K}^{-n}(X)$:

For $n \geq 0$ define $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$.

Define the unreduced version:

$$K^{-n}(X) = \tilde{K}^{-n}(X \sqcup pt)$$

Goals:

① Extend this definition to $K^n(X)$ for all $n \in \mathbb{N}$.

② Define relative K-theory $K(X, A)$ for $A \subset X$. Explain what it means for K-theory to be a generalised cohomology theory.

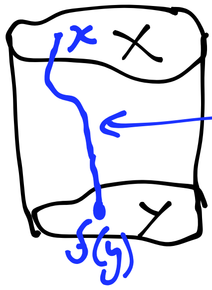
We do ② first.

More background:

Mapping cylinder & mapping cone

For $f: X \to Y$ define the mapping cylinder

$$M_f = \frac{(X \times I) \sqcup Y}{(x,1) \sim f(x)}$$

i.e. $M_f =$ 

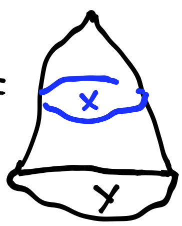
path $\gamma(i) = (x,i)$ links $x \in X$ to $f(x) \in Y$.

NB. The inclusion $Y \hookrightarrow M_f$ is a homotopy equivalence.

So up to homotopy equivalence every map $f: X \to Y$ is an inclusion $X \hookrightarrow M_f$:

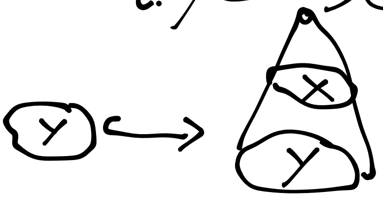
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, "\cong"] \\ X \arrow[r, hook] & M_f \end{tikzcd}$$

Define the mapping cone C_f to be the space obtained from M_f by identifying $X \times \{0\}$ to a single point.

i.e. $C_f =$ 

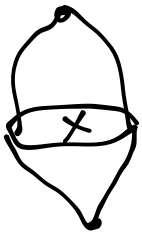
<u>Example:</u> What is the mapping cone of the inclusion $i : Y \hookrightarrow C_f$?

The map



has mapping cylinder:

So mapping cone is



$= SX$

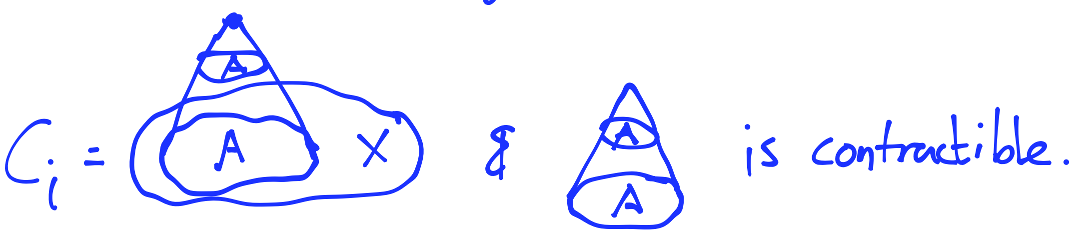
<u>Prop.</u> For $A \subset X$; $H^n(X,A) \simeq \tilde{H}^n(C_i)$, where $i : A \hookrightarrow X$ is the inclusion.

<u>proof.</u>

$$\tilde{H}^n(C_i) := H^n(C_i, *) \simeq H^n(X,A)$$

by excision

since



This proof just relied on axioms of generalised cohomology theory. So this tells us we should define

$$K^n(X,A) = \tilde{K}^n(C_i) \text{ where } i : A \hookrightarrow X \text{ is the inclusion.}$$

NB. $K(X,\emptyset) = K(X)$

Bott Periodicity

$\boxed{\text{Thm. There are natural isomorphisms } K^{-n-2}(X) \cong K^{-n}(X) \text{ for any compact top. space } X}$

By Bott Periodicity it makes sense to define $K^n(X)$ for any $n \in \mathbb{N}$ as:

$$K^n(X) = K^{-(n \bmod 2)}(X).$$

Thm. This definition gives a generalised cohomology theory.

Example of Bott Periodicity:

$$K^{-2n}(pt) = K(S^{2n}) = \mathbb{Z}$$
$$K^{-2n-1}(pt) = K(S^{2n+1}) = 0$$

NB. complex vector bundles over odd spheres are stably trivial, but are not all trivial.