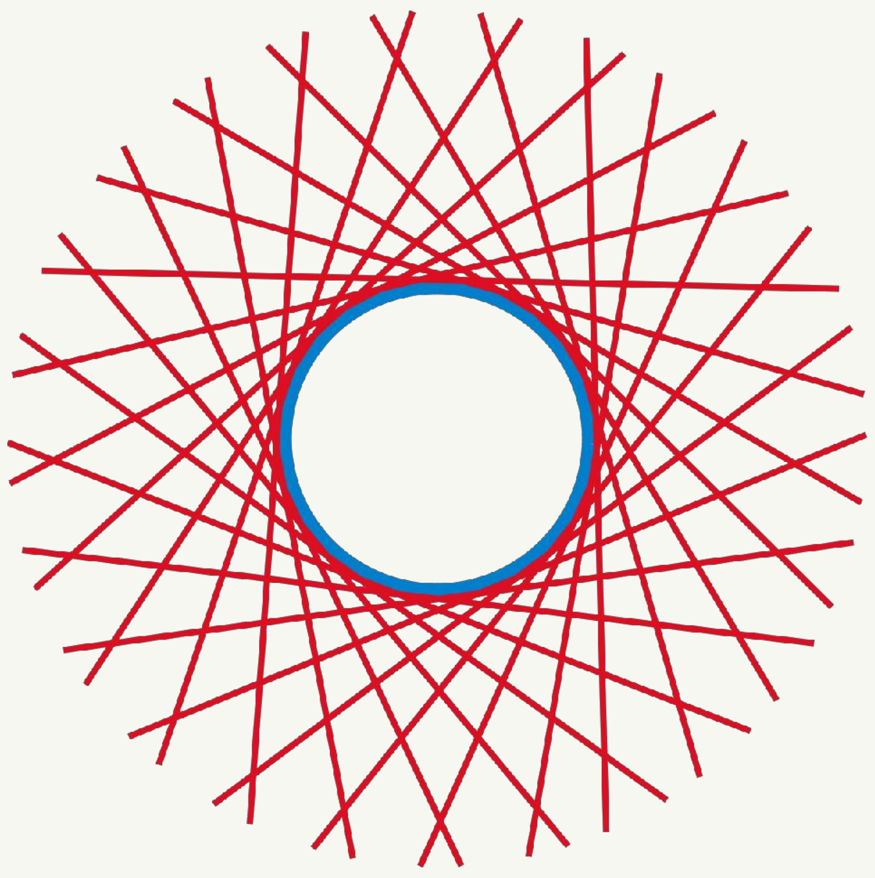
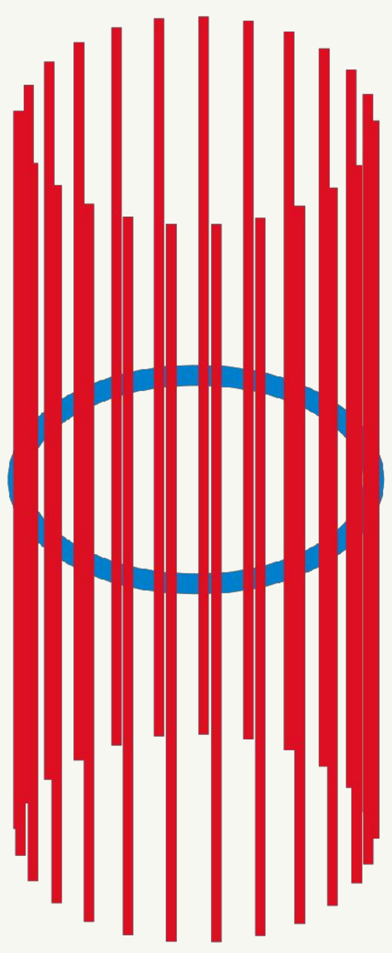
Definition

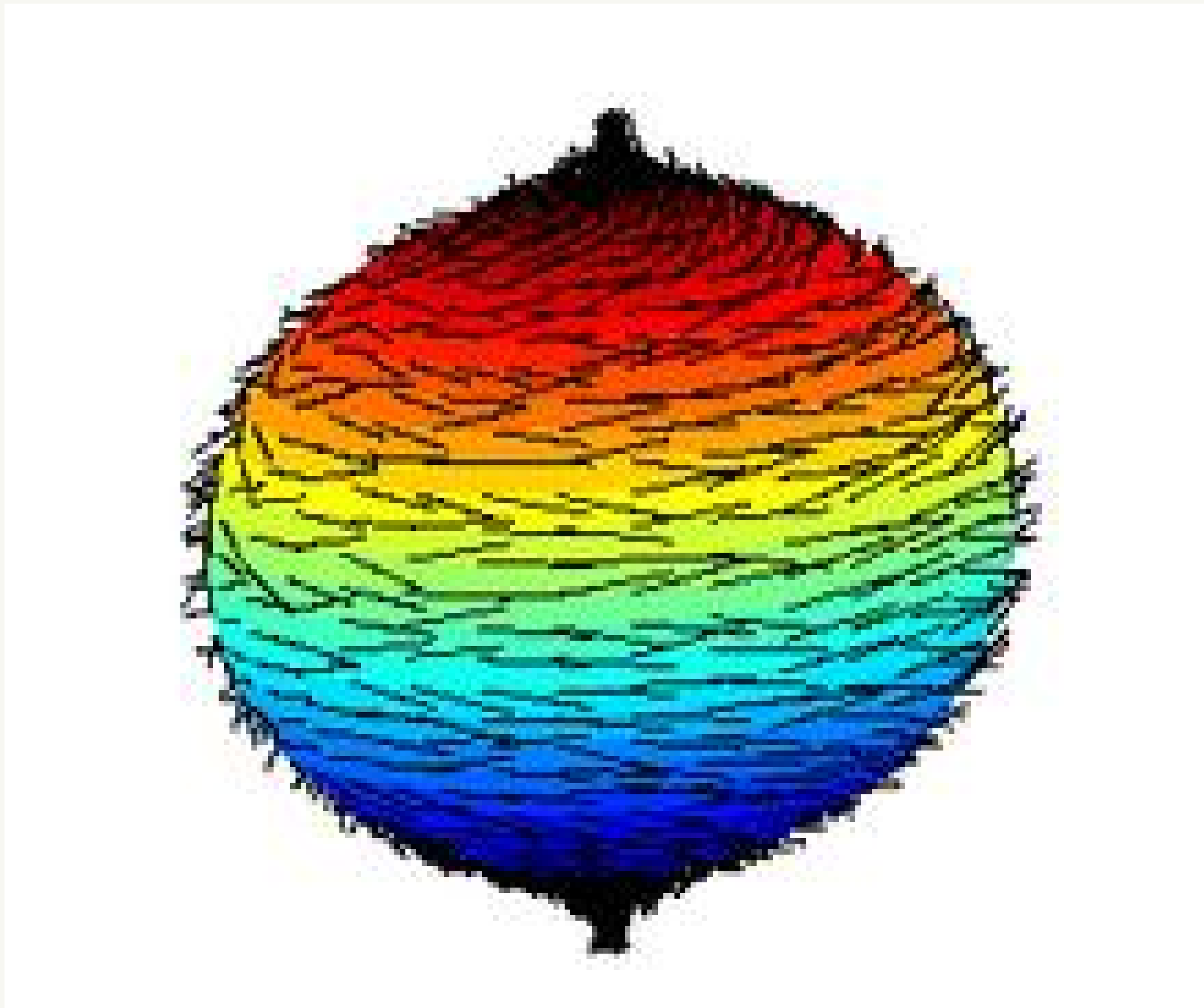
a sphere is parallelisable if the tangent bundle is trivial.

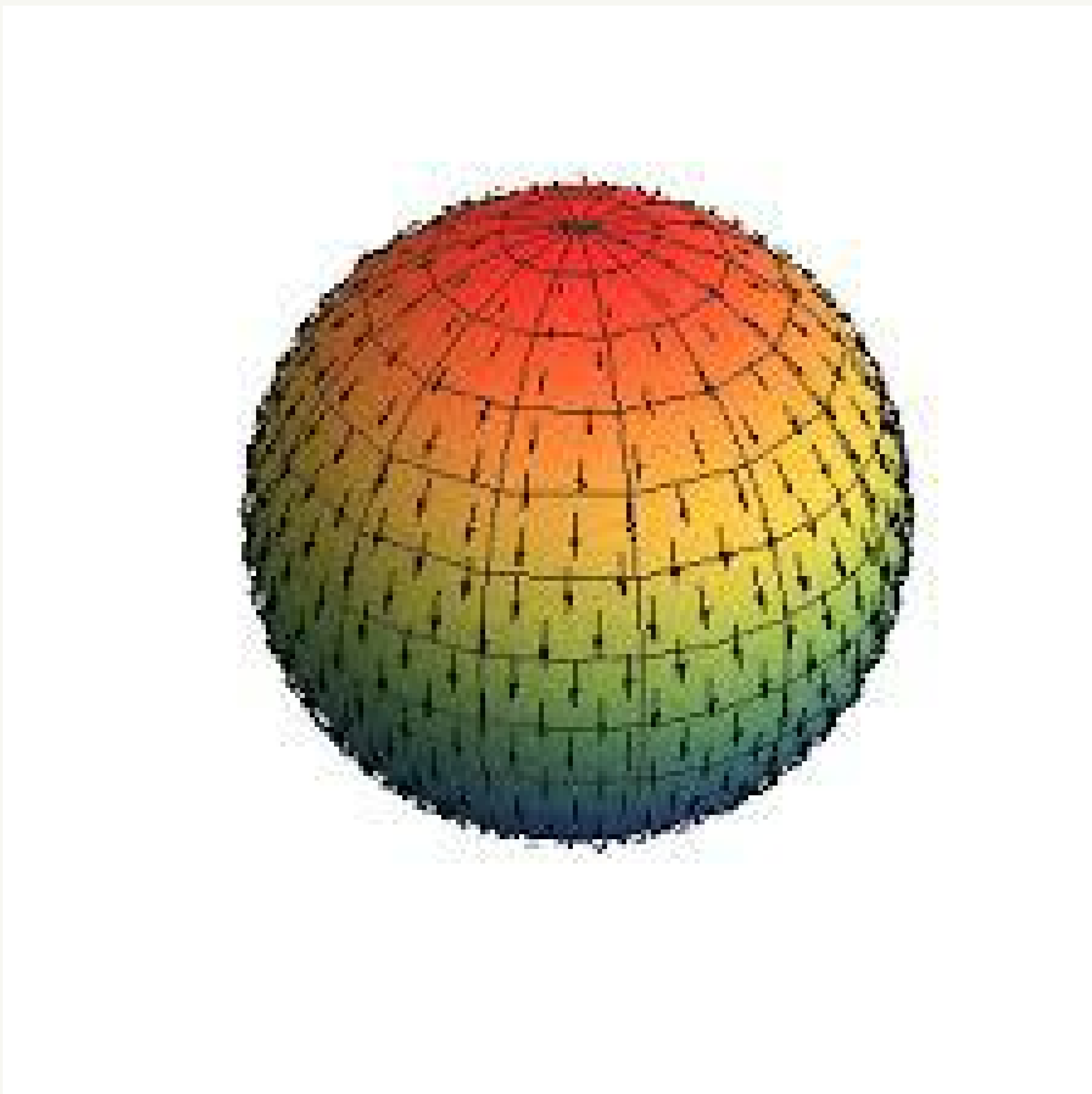


is



S^1 is parallelisable





S^2 is not!

Recall that an n-dimensional bundle

$p: E \to B$ is trivial iff

$\exists$ sections $s_i : B \to E$ $\quad (i=1,\dots,n)$

such that $s_1(b), \dots, s_n(b)$ are lin. ind

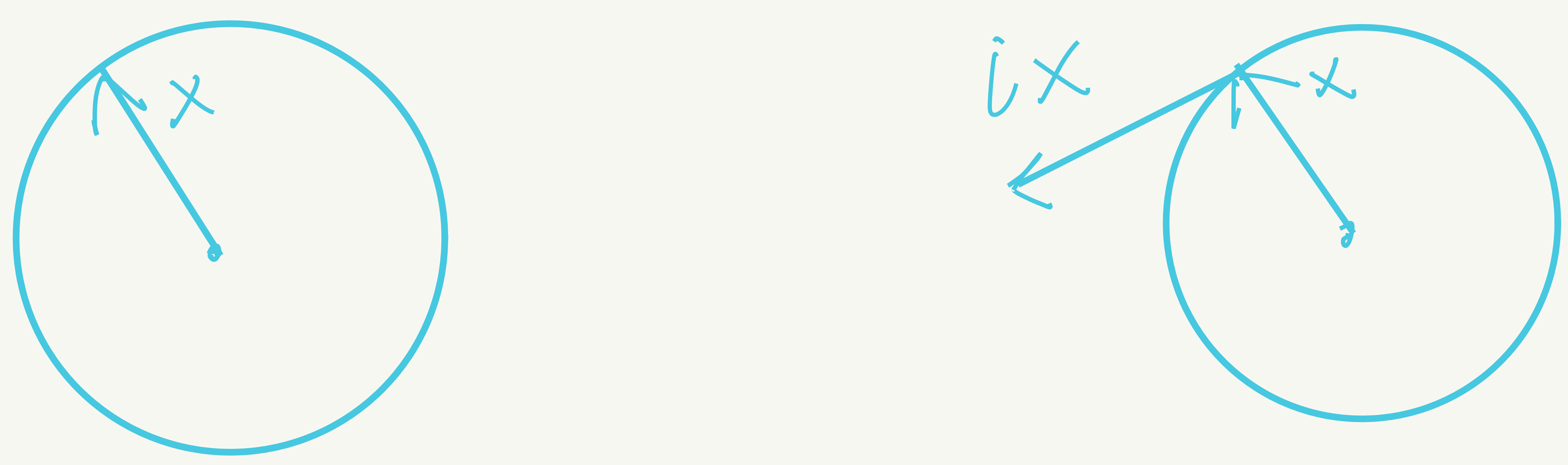
in the fiber $p^{-1}(b)$.

<u>Example</u> the tangent bundle to S^1

is trivial because it has section

$$s : \underset{\mathbb{R}^2 \simeq \mathbb{C}}{\overset{}{S^1 \cap}} \longrightarrow TS^1 \; ; \quad x \longmapsto (x, ix)$$

$\mathbb{R}^2 \simeq \mathbb{C} \to$ division algebra !



Example Josh showed S^2 is not parallelisable.

Example What about S^3 ?

Think of $\mathbb{R}^4$ as the quaternions $\mathbb{H}$

division algebra

$$\mathbb{R}^4 \cong \mathbb{H} = \langle 1, i, j, k \rangle_{\mathbb{R}} \quad \text{with} \quad i^2 = j^2 = k^2 = -1$$

$$ij = k \quad jk = i \quad ki = j$$

$$ji = -k \quad kj = -i \quad ik = -j$$

Then $|xy| = |x|\,|y|$ for all $x, y \in \mathbb{R}^4$.

Since $\{1, i, j, k\}$ is an orthonormal basis for $\mathbb{R}^4$, we get an orthonormal basis $\{x, xi, xj, xk\}$ $\forall x \in S^3 \subseteq \mathbb{R}^4$.

define sections

$s_1 : S^3 \longrightarrow TS^3 : x \longmapsto (x, ix)$

$s_2 : S^3 \longrightarrow TS^3 : x \longmapsto (x, jx)$

$s_3 : S^3 \longrightarrow TS^3 : x \longmapsto (x, kx)$

So yes S^3 is parallelisable!

x, ix, jx, kx are orthonormal basis!

Example $\quad S^7 \subseteq \mathbb{R}^8 \approx \mathbb{O}$

$\mathbb{R}^8 \approx \mathbb{O} = \langle 1, i, j, k, E, I, J, K \rangle_{\mathbb{R}}$

with multiplication table

	1	i	j	k	E	I	J	K
1	1	i	j	k	E	I	J	K
i	i	-1	$-k$	j	I	$-E$	$-K$	J
j	j	k	-1	$-i$	J	K	$-E$	$-I$
k	k	$-j$	i	-1	K	$-J$	I	$-E$
E	E	$-I$	$-J$	$-K$	-1	i	j	k
I	I	E	$-K$	J	$-i$	-1	$-k$	j
J	J	K	E	$-I$	$-j$	k	-1	$-i$
K	K	$-J$	I	E	$-k$	$-j$	i	-1

(again $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}^8$)

$\rightarrow$ can construct 7 orthogonal tangent fields on S^7

$\rightarrow$ S^7 is parallelisable

Definition A ***division algebra*** A is an algebra over a field in which you can "divide": $\forall x, y \in A,\ y \neq 0: \ \exists z_1: \ x = z_1 y$

$\exists z_2: \ x = y z_2$

not nec. associative or unital

Remark can always change the multiplication a bit to create a unit e.
Over $\mathbb{R}$, can create unit with $|e| = 1$

Examples of finite dim. division algebras / $\mathbb{R}$:

- $\mathbb{R}$ dim 1
- $\mathbb{C}$ dim 2
- $\mathbb{H}$ dim 4
- $\mathbb{O}$ dim 8
- ... ?

Theorem The following are only true for $n = 1, 2, 4, 8$:

(a) $\mathbb{R}^n$ is a division algebra

(b) S^{n-1} is parallelisable

Proof (first part) We already know (a) and (b) hold when $n = 1, 2, 4, 8$.

Definition We say S^{n-1} is an H-space if there exists a mult. map $S^{n-1} \times S^{n-1} \longrightarrow S^n$ with a 2-sided identity.

Lemma If (a) or (b) hold then S^{n-1} is an H-space.

Proof If (a) holds then define

$$S^{n-1} \times S^{n-1} \longrightarrow S^{n-1} : (x, y) \longmapsto \frac{xy}{|xy|}.$$

If (b) holds,

$$\begin{tikzcd} S^{n-1} \times \mathbb{R}^{n-1} \arrow[r, "\sim"] \arrow[dr] & TS^{n-1} \arrow[d] \\ & S^{n-1} \end{tikzcd}$$

can choose sections $s_1, \dots, s_{n-1}$ such that

$$s_1(e_1), \dots, s_{n-1}(e_1) = e_2, e_3, \dots, e_n$$

and $s_1(x), \dots, s_{n-1}(x)$ orthonormal $\forall x \in S^{n-1}$.

Define $A_x := \begin{bmatrix} | & | & & | \\ s_1(x) & s_2(x) & \cdots & s_{n-1}(x) \\ | & | & & | \end{bmatrix}.$

Then $S^{n-1} \times S^{n-1} \longrightarrow S^{n-1} : (x, y) \longmapsto A_x y$

defines an H-space structure with unit e_1.

$\square$

"Reminder"

① $\mathbb{Z}[H]/(H-1)^2 \xrightarrow{\sim} K(S^2)$

where H is the canonical line bundle over $S^2 = \mathbb{C}P^1$.

② $\tilde{K}(S^2) \hookrightarrow K(S^2) \longrightarrow K(x_0)$

$\tilde{K}(S^2) = \langle H-1 \rangle_{\mathbb{Z}} \cong \mathbb{Z}$ with trivial multiplication

③ $\tilde{K}(S^n) = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z} = \langle (H-1) * \cdots * (H-1) \rangle & n \text{ even} \end{cases}$

④ $K(S^{2k}) \otimes K(X) \xrightarrow{\sim} K(S^{2k} \times X)$

⑤ $\tilde{K}(S^{2k}) \otimes \tilde{K}(X) \xrightarrow{\sim} \tilde{K}(S^{2k} \wedge X)$

Bott periodicity

We will show

$n \notin \{1,2,4,8\} \Longrightarrow S^{n-1}$ not an H-space

Proof (when n is odd) $\boxed{n-1=2k}$

Suppose S^{2k} is an H-space.

$$\mu : S^{2k} \times S^{2k} \longrightarrow S^{2k}$$

induces $K(S^{2k}) \xrightarrow{\mu^*} K(S^{2k} \times S^{2k})$

$$K(S^{2k}) \cong \mathbb{Z}[\gamma]/(\gamma^2)$$

$$K(S^{2k} \times S^{2k}) \overset{(4)}{\cong} K(S^{2k}) \otimes K(S^{2k}) \cong \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2)$$

$$S^{2k} \overset{i}{\underset{j}{\rightrightarrows}} S^{2k} \times S^{2k} \xrightarrow{\mu} S^{2k} \qquad = id$$

$$x \longmapsto (e, x)$$

$$x \longmapsto (x, e)$$

$$i^*: \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2) \longrightarrow \mathbb{Z}[\gamma]/(\gamma^2)$$

$$\alpha \longmapsto 0$$

$$\beta \longmapsto \gamma$$

$$j^*: \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2) \longrightarrow \mathbb{Z}[\gamma]/(\gamma^2)$$

$$\beta \longmapsto 0$$

$$\alpha \longmapsto \gamma$$

So

$$
\mathbb{Z}[\gamma]/(\gamma^2) \xrightarrow{\mu^*} \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2) \underset{j^*}{\overset{i^*}{\rightrightarrows}} \mathbb{Z}[\gamma]/(\gamma^2)
$$

$$
\gamma \overset{\mu^*}{\longmapsto} n + a\alpha + b\beta + m\alpha\beta \;
\begin{array}{l}
\overset{i^*}{\longmapsto} n + a\gamma \\[1em]
\underset{j^*}{\longmapsto} n + b\gamma
\end{array}
$$

shows $n = 0$, $\quad a = b = 1$.

So $\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$.

But then

$$
0 = \mu^*(\gamma^2) = (\alpha + \beta + m\alpha\beta)^2
$$

$$
= 2\alpha\beta \neq 0
$$

↯

From now on, we assume $n = 2k$ even.

We want to show S^{2k-1} is <u>not</u>

an H-space when $n = 2k \notin \{2, 4, 8\}$.

or $k \notin \{1, 2, 4\}$.

Definition Given a map $f: S^{4k-1} \to S^{2k}$, we can attach a cell e^{4k} to S^{2k} via f to get a top. space C_f with $C_f / S^{2k} \approx S^{4k}$.

So we get a SES:

$$\begin{array}{c} \tilde{K}^1(S^{2k}) \\ \| \\ 0 \end{array} \to \tilde{K}(S^{4n}) \longrightarrow \tilde{K}(C_f) \longrightarrow \tilde{K}(S^{2k}) \to 0$$

$$(H-1) * \cdots * (H-1) \longmapsto \alpha$$

$$\beta \longmapsto (H-1) * \cdots * (H-1)$$

Define α, choose β as above.

Since $\beta^2 \mapsto 0$, we know $\beta^2 = h\alpha$ $\exists h$.

h is the Hopf invariant of f.

Exercise: h does not depend on choice of β.

Suppose S^{2k-1} is an H-space:

$$\mu: S^{2k-1} \times S^{2k-1} \longrightarrow S^{2k-1}.$$

Define an extension

$$\begin{array}{ccc} \hat{\mu}: \; S^{4k-1} & \longrightarrow & S^{2k} \\ \| & & \| \\ \partial(D^{2k} \times D^{2k}) & & D^{2k}_{+} \cup_{S^{2k-1}} D^{2k}_{-} \\ \| & & \\ \partial D^{2k} \times D^{2k} \cup_{S^{2k-1} \times S^{2k-1}} D^{2k} \times \partial D^{2k} & & \end{array}$$

by

$$\partial D^{2k} \times D^{2k} \xrightarrow{\hat{\mu}} D^{2k}_{+} : (x,y) \longmapsto |y| \, \mu\left(x, \frac{y}{|y|}\right)$$

$$D^{2k} \times \partial D^{2k} \xrightarrow{\hat{\mu}} D^{2k}_{-} : (x,y) \longmapsto |x| \, \mu\left(\frac{x}{|x|}, y\right)$$

- well defined ✓
- continuous ✓
- extends μ ✓

Summary so far:

$\mu : S^{2k-1} \times S^{2k-1} \longrightarrow S^{2k-1}$

$\hat{\mu} : S^{4k-1} \longrightarrow S^{2k}$

$C_{\hat{\mu}}$: e^{4k} attached to S^{2k} via $\hat{\mu}$

SES : $0 \to \tilde{K}(S^{4k}) \longrightarrow \tilde{K}(C_{\hat{\mu}}) \longrightarrow \tilde{K}(S^{2}) \to 0$

Lemma If μ is an H-space multiplication, then $\hat{\mu} : S^{4k-1} \longrightarrow S^{2k}$ has Hopf invariant 1, -1.

Theorem < Adams > There exists a map $S^{4k-1} \longrightarrow S^{2k}$ of odd Hopf invariant only when $k = 1, 2$ or 4.

ADAMS OPERATIONS

use splitting principle to prove this... maybe future seminar?

Theorem There exist ring morphisms

$$\psi^n : K(X) \longrightarrow K(X) \quad \text{for all } n \geq 0$$

such that

(1) $\psi^n f^* = f^* \psi^n \quad \forall f : X \to Y$

("naturality")

(2) $\psi^n(L) = L^n$ if L is a line bundle

(3) $\psi^n \circ \psi^l = \psi^{nl}$

(4) $\psi^p(\alpha) \equiv \alpha^p \mod p$

Remark By **naturality**, also get

$$\psi^n : \tilde{K}(X) \longrightarrow \tilde{K}(X)$$

$$\begin{array}{c} \tilde{K}(X) \\ \downarrow \\ K(X) \\ \downarrow \\ K(x_0) \end{array}$$

Example $\psi^n : \tilde{K}(S^{2k}) \longrightarrow \tilde{K}(S^{2k})$

$$\begin{tikzcd} \tilde{K}(S^{2k}) \arrow[r, "\psi^n"] \arrow[d, phantom, "\cong"] & \tilde{K}(S^{2k}) \arrow[d, phantom, "\cong"] \\ \mathbb{Z} \arrow[r, "\cdot n^k"] & \mathbb{Z} \end{tikzcd}$$

$\mathbb{Z} = \langle H-1 \rangle$

$\boxed{k=1}$ $\psi^n(H-1) = H^n - 1 = (1 + (H-1))^n - 1$

$= 1 + n(H-1) - 1$

$= n(H-1)$ ✓

$\boxed{k>1}$ by induction

$$\tilde{K}(S^2) \otimes \tilde{K}(S^{2k-2}) \xrightarrow{\sim} \tilde{K}(S^{2k})$$

$\psi^n(\alpha * \beta) = \psi^n(\alpha) * \psi^n(\beta)$

$= n\alpha * n^{k-1}\beta$

$= n^k(\alpha * \beta)$. ✓

Proof of Adams theorem

There is a map $S^{4k-1} \longrightarrow S^{2k}$ with h odd only when $k=1, 2$ or 4.

$$0 \to \tilde{K}(S^{4k}) \longrightarrow \tilde{K}(C_{\tilde{u}}) \longrightarrow \tilde{K}(S^{2}) \to 0$$

$$(H-1)*\cdots*(H-1) \longmapsto \alpha$$

$$\beta \longmapsto (H-1)*\cdots*(H-1)$$

$$\psi^n(\alpha) = n^{2k}\alpha$$

$$\psi^n(\beta) = n^k\beta + \mu_n\alpha$$

$$\psi^n\psi^\ell(\beta) = \psi^n(\ell^k\beta + \mu_\ell\alpha) = n^k\ell^k\beta + (n^{2k}\mu_\ell + \ell^k\mu_n)\alpha$$

$$\psi^\ell\psi^n(\beta) = \quad \cdots \quad = \ell^k n^k\beta + (\ell^{2k}\mu_n + n^k\mu_\ell)\alpha.$$

so $n^{2k}\mu_\ell + \ell^k\mu_n = \ell^{2k}\mu_n + n^k\mu_\ell$

$$n^{2k}\mu_\ell + \ell^k \mu_n = \ell^{2k}\mu_n + n^k \mu_\ell$$

so $\boxed{(n^{2k} - n^k)\mu_\ell = (\ell^{2k} - \ell^k)\mu_n}$.

Since $\psi^2(\beta) \equiv_2 \beta^2 = h\alpha$,

and $\psi^2(\beta) = 2^k\beta + \mu_2\alpha$, we see $\boxed{\mu_2 \equiv_2 h}$.

Take $n=2$ and $\ell=3$:

$\boxed{(2^{2k} - 2^k)\mu_3 = (3^{2k} - 3^k)\mu_2}$, so

2^k divides $3^k(3^k - 1)\mu_2$

If h is odd, then μ_2 is odd, so 2^k divides 3^{k-1}.

After fiddling around, this shows

$k = 1, 2$ or 4.