K-THEORY LECTURE 6:

Grassmannians and the universal bundle

Main goal:

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, define an n-dimensional $\mathbb{K}$-vector bundle

$$\xi : E_n(\mathbb{K}^\infty) \longrightarrow G_n(\mathbb{K}^\infty)$$

with the property that for any paracompact space X the map

$$[X, G_n(\mathbb{K}^\infty)] \longrightarrow \mathrm{Vect}^n_{\mathbb{K}}(X)$$
$$f \longmapsto f^*(\xi)$$

homotopy classes of maps $X \to G_n(\mathbb{K}^\infty)$

n-dim'l $\mathbb{K}$-vector spaces over X

is a bijection

i.e every vector bundle over X is the pullback of the universal bundle $\xi : E_n(\mathbb{K}^\infty) \longrightarrow G_n(\mathbb{K}^\infty)$.

I. Review of paracompact spaces:

For our purposes, a space X is paracompact if every open cover of X admits a partition of unity:

i.e. If $X = \bigcup_{i\in I} U_i$ is an open cover, then there are continuous functions $\varphi_i : X \to [0,1]$ for all $i \in I$ such that $\operatorname{supp}\varphi_i \subseteq U_i$,

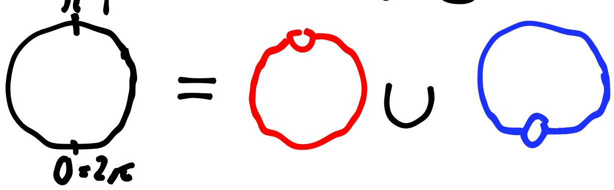
$$\operatorname{supp}\varphi_i = \{x \in X \mid \varphi_i(x) \neq 0\}$$

and every $x \in X$ has an open neighbourhood $V \ni x$ such that:

(i) $\varphi_i|_V \neq 0$ for finitely many $i \in I$.

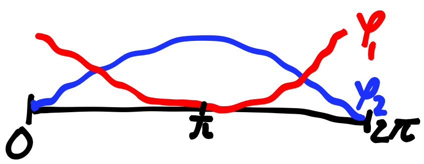
(ii) $\sum_{i\in I} \varphi_i(v) = 1$ for all $v \in V$.

E.g. consider the open cover of S^1:



Define $\varphi_1, \varphi_2 : [0, 2\pi) \longrightarrow [0,1]$ by

$$\varphi_1(x) = \cos^2\left(\tfrac{1}{2}x\right)$$
$$\varphi_2(x) = \sin^2\left(\tfrac{1}{2}x\right)$$



This gives a partition of unity for the cover of S^1.

NB. Sometimes a weaker definition of paracompact is used and what we call paracompact would be equivalent to paracompact + Hausdorff

Some sufficient conditions for paracompactness:

(i) CW-complex $\Rightarrow$ paracompactness

(ii) metric space $\Rightarrow$ paracompact

(iii) compact Hausdorff $\Rightarrow$ paracompact

II. Grassmannian varieties

Define the space $G_n(\mathbb{K}^m) = \{n\text{-dimensional subspaces of } \mathbb{K}^m\}$ topologised as a quotient of the Stiefel variety:

$$\begin{array}{c} V_n(\mathbb{K}^m) = \{\text{ordered } n\text{-tuples of orthonormal vectors in } \mathbb{K}^m\} \\ \downarrow \\ (\mathbb{K}^m)^n \end{array}$$

Here $V_n(\mathbb{K}^m) \twoheadrightarrow G_n(\mathbb{K}^m)$ by taking span

$\boxed{\text{Prop. } G_n(\mathbb{K}^m) \text{ is paracompact.}}$

Two proofs:

(1) $G_n(\mathbb{K}^m)$ is a CW-complex by Schubert calculus.

(2) $V_n(\mathbb{K}^m)$ is compact since it is a subspace of $(S^m)^n$

So $G_n(\mathbb{K}^m)$ is compact

Defn. Tautological bundle of $G_n(\mathbb{K}^m)$.

$$\begin{tikzcd} (L,v) \arrow[d, mapsto] & E_n(\mathbb{K}^m) = \{(L,v)\in G_n(\mathbb{K}^m)\times\mathbb{K}^m \mid v\in L\} \arrow[d] \\ L & G_n(\mathbb{K}^m) \end{tikzcd}$$

Exercise: This is a rank n vector bundle.

Defn. The universal vector bundle.

The inclusions $\cdots \hookrightarrow \mathbb{K}^m \hookrightarrow \mathbb{K}^{m+1} \hookrightarrow \cdots$

induce inclusions $\cdots \hookrightarrow G_n(\mathbb{K}^m) \hookrightarrow G_n(\mathbb{K}^{m+1}) \hookrightarrow \cdots$

$\cdots \hookrightarrow E_n(\mathbb{K}^m) \hookrightarrow E_n(\mathbb{K}^{m+1}) \hookrightarrow \cdots$

and so we define $G_n(\mathbb{K}^\infty) := \bigcup_m G_n(\mathbb{K}^m)$, $E_n(\mathbb{K}^\infty) = \bigcup_m E_n(\mathbb{K}^m)$

with the direct limit topology. i.e. $U \subseteq G_n(\mathbb{K}^\infty)$ is open iff $U \cap G_n(\mathbb{K}^m)$ is open for each m

Define the universal bundle:

$$\xi : E_n(\mathbb{K}^\infty) \to G_n(\mathbb{K}^\infty)$$
$$(L,v) \longmapsto L$$

Henceforth write $E_n := E_n(\mathbb{K}^\infty)$ and $G_n := G_n(\mathbb{K}^\infty)$.

Theorem: For a paracompact space X, the map

$$[X, G_n] \to \mathrm{Vect}^n(X)$$
$$f \longmapsto f^*(\xi)$$

is a bijection.

Before giving the proof we begin with an example:

Consider the tangent bundle:
$$\begin{tikzcd} TS^n := \{(x,v)\in S^n\times\mathbb{R}^{n+1} \mid x\perp v\} \arrow[d] \\ S^n \end{tikzcd}$$

Each fibre $p^{-1}(x)$ is a point in $G_n(\mathbb{R}^{n+1})$ and so we have a map

$$f: S^n \longrightarrow G_n(\mathbb{R}^{n+1}) \hookrightarrow G_n(\mathbb{R}^{\infty})$$
$$x \longmapsto p^{-1}(x)$$

Now consider the pullback:

$$\begin{tikzcd} f^*(\xi) \arrow[r] \arrow[d] & E_n \arrow[d, "\xi"] & (L,v) \arrow[d, mapsto] \\ S^n \arrow[r, "f"] & G_n & L \\ x \arrow[r, mapsto] & p^{-1}(x) & \end{tikzcd}$$

$\rightsquigarrow$
$$f^*(\xi) = \{(x,L,v)\in S^n\times E_n \mid L = p^{-1}(x)\}$$
$$\simeq \{(x,v) \mid v\in p^{-1}(x)\}$$
$$\simeq TS^n$$

Proof of Theorem:

Lemma: For a n-dim vector bundle $p: E\to X$ there is a bijection:

$$\{f: X\to G_n \mid E\simeq f^*(\xi)\} \longleftrightarrow \left\{\begin{array}{l}\text{maps } g: E\to \mathbb{K}^{\infty} \\ \text{that are a linear injection} \\ \text{on each fibre.}\end{array}\right\}$$

i.e. for any $x\in X$, $g|_{p^{-1}(x)}: p^{-1}(x)\longrightarrow \mathbb{R}^{\infty}$ is a linear injection

proof of lemma:

($\Rightarrow$) Given a map $f: X\to G_n$ and an isomorphism $E\simeq f^*(\xi)$ we have a commutative diagram:

$$\begin{tikzcd} & (x,f(x),v) \arrow[r, mapsto] & (f(x),v) \arrow[r, mapsto] & v \\ E \arrow[r, "\sim"] \arrow[dr, "p"'] & f^*(\xi) \arrow[r] \arrow[d, "\tilde{p}"] & E_n \arrow[r] \arrow[d] & \mathbb{K}^{\infty} \\ & X \arrow[r, "f"] & G_n & \end{tikzcd}$$

$(L,v)\longmapsto v$, $\quad L$

Then the composition $g: f^*(\xi)\longrightarrow E_n \to \mathbb{K}^{\infty}$ is a linear injection on each fibre.

i.e. $g|_{\tilde{p}^{-1}(x)}: \tilde{p}^{-1}(x) \simeq \{v \mid v\in f(x)\} \hookrightarrow \mathbb{K}^{\infty}$

($\Leftarrow$) Given $g: E\to\mathbb{K}^{\infty}$ define $f: X\longrightarrow G_n$ by $x\mapsto g(p^{-1}(x))$ $\longleftarrow$ since g is injective on fibres this is an n-dim'l subspace of $\mathbb{R}^{\infty}$

EXERCISE: Show the operations are mutually inverse.

proof of theorem continued...

(1) Surjectivity of $[X, G_n] \longrightarrow \mathrm{Vect}^n(X)$

Suppose $p: E \to X$ is a rank n vector bundle.

X paracompact $\Rightarrow$ there is a countable open cover $X = \bigcup_{i \in I} U_i$ such that E is trivial over each U_i.

Let $\{\varphi_i\}$ be a partition of unity for this cover.

Aim: define a function $g: E \to \mathbb{K}^\infty$ that is a linear injection on each fibre.

Let $\tilde{g}_i : p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{K}^n \xrightarrow{\text{proj}} \mathbb{K}^n$

Then consider the map $p^{-1}(U_i) \longrightarrow \mathbb{K}^n : v \mapsto \varphi_i(p(v)) \cdot g_i(v)$

i.e. $\tilde{g}_i$ "scaled by φ_i".

Since $\operatorname{supp} \varphi_i \subseteq U_i$ this extends to a continuous map (extending by zero)

$$g_i : E \longrightarrow \mathbb{K}^n.$$

These g_i form the co-ordinates of the required map:

$$g: E \longrightarrow (\mathbb{K}^n)^I \hookrightarrow \mathbb{K}^\infty$$

Since I is countable

(2) INJECTIVITY OF $[X, G_n] \longrightarrow \mathrm{Vect}^n(X)$

idea: Show that if we have two isomorphisms $E \xrightarrow{\sim} f_0^*(\xi)$ and $E \xrightarrow{\sim} f_1^*(E_n)$ for two maps $f_0, f_1 : X \to G_n$ then the resulting maps $g_0, g_1 : E \to \mathbb{R}^\infty$ are homotopic through maps g_t that are linear injections on fibres.

III. Application to K-theory:

Consider $\cdots \hookrightarrow G^n(\mathbb{C}^\infty) \hookrightarrow G^{n+1}(\mathbb{C}^\infty) \hookrightarrow \cdots$

Define the direct limit topological space $BU := \bigcup_n G_n(\mathbb{C}^\infty)$

Then
$$[X, BU] \simeq \left\{ \begin{matrix} \text{complex vector} \\ \text{bundles over} \\ X \end{matrix} \right\} \Big/ \sim$$

where $E \sim F$ if $\exists$ trivial bundles e_k, e_l s.t. $E \oplus e_k \simeq E \oplus e_l$

This is exactly Joe's definition of reduced K-theory. i.e. $\widetilde{K}(X) = [X, BU]$ if X is compact.

<u>Exercise</u>: For X compact, $K(X) = [X, BU_{\times\mathbb{Z}}]$.

This gives a natural way to extend the definition of K-theory to all paracompact spaces

$$K(X) := [X, BU_{\times\mathbb{Z}}]$$

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