SAS 30/09/20

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Leray - Hirsch theorem: R - Some coefficient ring

Definition: Let $\varphi \in C^p(X,R)$ and $\psi \in C^q(X,R)$. Then the cup product $\varphi \smile \psi \in C^{p+q}(X,R)$ is defined by its value on $\sigma : \Delta^{p+q} \longrightarrow X$

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[1,\dots,p]}) \cdot \psi(\sigma|_{[p+1,\dots,p+q]})$$

Proposition: The cup product on cochains induces a product on singular cohomology $\smile : H^p(X,R) \times H^q(X,R) \longrightarrow H^{p+q}(X,R)$
(Exercise: check that $\varphi \smile \psi \in [\varphi \smile \psi]$)

Remark: The cup product induces the wedge product in de Rham cohomology.

Theorem (Leray-Hirsch): Let $\pi : E \longrightarrow X$ be a fibre bundle over paracompact X. Suppose that there exist $x_1, \dots, x_k \in H^\bullet(E,R)$ such that their restrictions $i_b^* x_1, \dots, i_b^* x_k$ form a basis of $H^\bullet(F,R)$ where $i_b : F \longrightarrow \pi^{-1}(b) \subset E$ is the inclusion. Then $H^\bullet(E,R)$ is a free $H^\bullet(X,R)$-module with basis $x_1, \dots, x_k$.

Given $\alpha \in H^\bullet(X,R)$, $\beta \in H^\bullet(E,R)$ then

$$\alpha\beta = \pi^*(\alpha) \smile \beta .$$

Theorem: Given a real vector bundle $\pi : E \rightarrow X$, there exist unique classes $w_i(E) \in H^i(X, \mathbb{Z}_2)$ such that

a) if $f : Y \rightarrow X$, then $w_i(f^*(E)) = f^*(w_i(E))$

b) If $\pi' : F \rightarrow X$ is another v.b, then

$$w(E \oplus F) = w(E) \smile w(F)$$

where $w(E) = 1 + w_1(E) + w_2(E) + \cdots$

c) $w_i(E) = 0 \quad \forall i > \dim(E)$

d) $w_1(E)$ is a generator of of $H^1(\mathbb{R}P^\infty, \mathbb{Z}_2)$ the canonical line bundle $\to \mathbb{R}P^\infty$

d) $w_1(E)$ is a generator of of $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$ the canonical line bundle over $\mathbb{RP}^\infty$.

Theorem: Given a complex vector bundle $\pi: E \to X$, there exist unique classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ such that

a) if $f: Y \to X$, then $c_i(f^*(E)) = f^*(c_i(E))$

b) If $\pi': F \to X$ is another v.b, then

$$c(E \oplus F) = c(E) \smile c(F)$$

where $c(E) = 1 + c_1(E) + c_2(E) + \cdots$

c) $c_i(E) = 0 \quad \forall i > \dim(E)$

d) $c_1(E)$ is a generator of the canonical line bundle over $\mathbb{CP}^\infty$ specified in advance.

Proof: Existence Consider the projectivised bundle of E (n dimensional v.b)

$$P(E) = E/\{v \sim \lambda v \text{ for } v \in \pi^{-1}(b)\ \forall b \in X\}$$

$$\begin{array}{c} P(E) \\ \big\downarrow {\scriptstyle P(\pi)} \\ X \end{array} \qquad \text{Has fibre } \mathbb{RP}^{n-1}$$

Idea is to apply Leray-Hirsch to $P(E)$. Need to find Need to find $x_i \in H^i(P(E), \mathbb{Z}_2)$ that restrict to generators of $H^i(\mathbb{RP}^{n-1}, \mathbb{Z}_2)$ in each fibre for $i = 0, \cdots, n-1$.

Recall that for any v.b $E \to X$ there exists $f: X \to G_n(\mathbb{R}^\infty)$ s.t $E = f^*(E_n(\mathbb{R}^\infty))$

$$\begin{array}{ccc} E & \xrightarrow{f^*} & E_n(\mathbb{R}^\infty) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & G_n(\mathbb{R}^\infty) \end{array}$$

Equivalently there exists a map $g: E \to \mathbb{R}^\infty$ that is a linear injection on the fibres.

Projectivise g to $P(g): P(E) \to \mathbb{RP}^\infty$

i.e

$$\begin{array}{ccc} E & \xrightarrow{g} & \mathbb{R}^\infty \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ P(E) & \xrightarrow{P(g)} & \mathbb{RP}^\infty \end{array}$$

Then $P(g)$ is an embedding on each fibre.

Facts: $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$ and

$$H^*(\mathbb{RP}^{n-1}; \mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^n)$$

Then $P(g)$ is an embedding on each fibre.

Facts: $H^\bullet(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$ and

$H^\bullet(\mathbb{RP}^{n-1}, \mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^n)$

Let α be a generator of $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$. Then $\alpha^i \in H^i(\mathbb{RP}^\infty, \mathbb{Z}_2)$.

$$\underset{\text{inclusion}}{\mathbb{RP}^{n-1} \xhookrightarrow{i} P(E)} \underset{\text{embedding}}{\xrightarrow{P(g)}} \mathbb{RP}^\infty$$

The composition $P(g)\circ i$ is an inclusion $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^\infty$ so α pulls back to β.

Thus $x^i = P(g)^*(\alpha^i)$ for $i = 0, \cdots, n-1$ give the desired generators of $H^i(X, \mathbb{Z}_2)$. These are independent of g since from a previous lecture all linear injections $g: E \to \mathbb{R}^\infty$ are homotopic and thus induce the same pullback.

Then by Leray-Hirsch, $H^\bullet(P(E), \mathbb{Z}_2)$ is a free $H^\bullet(X, \mathbb{Z}_2)$-module with basis $\{1, x, \cdots, x^{n-1}\}$. So x^n can be written uniquely as

$$x^n + w_1(E)\,x^{n-1} + \cdots + w_n(E)\cdot 1 = 0$$

where $w_i(E)\,x^j = P(\pi)^*(w_i(E)) \smile x^j$

Now we need to prove that (a) — (d) hold for the coefficients $w_i(E)$.

(a) Let $f: Y \to X$

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

$\tilde{f}: f^*(E) \to E$ is an isomorphism on each fibre. (Hatcher prop^{n}1.5)

For this reason, $P(\tilde{f})$ takes the class $x = x(E)$ to $x(f^*(E))$

$$P(\tilde{f})^*(w_i(E)\,x(E)^j) = P(\tilde{f})^*(P(\pi)^*(w_i(E)) \smile x(E)^j)$$
$$= P(\tilde{f})^* \circ P(\pi)^*(w_i(E)) \smile P(\tilde{f})^*(x(E)^j)$$

$$\begin{array}{ccc} H^\bullet(f^*(E)) & \xleftarrow{P(\tilde{f})^*} & H^\bullet(E) \\ {\scriptstyle P(f^*\pi)^*}\uparrow & & \uparrow{\scriptstyle P(\pi)^*} \\ & f^* & \end{array}$$

$$\begin{array}{ccc} P(f^*\pi)^* \uparrow & & \uparrow P(\pi)^* \\ H^\cdot(Y) & \xleftarrow{f^*} & H^\cdot(X) \end{array}$$

$$= P(f^*\pi)^* \circ f^*(w_i(E)) \cup x(f^*E)^j$$
$$= f^*(w_i(E))\, x(f^*(E))^j$$

The relation $x^n + w_1(E)x^{n-1} + \cdots + w_n(E)\cdot 1 = 0$ pulls back to
$x(f^*(E))^n + f^*(w_1(E))\, x(f^*(E))^{n-1} + \cdots + f^*w_n(E)\cdot 1 = 0$

So by uniqueness of the relation we have $w(f^*(E)) = f^*w(E)$

(b)

The inclusions $E_1 \hookrightarrow E_1 \oplus E_2 \hookleftarrow E_2$ induce inclusions $P(E_1) \hookrightarrow P(E_1 \oplus E_2) \hookleftarrow P(E_2)$

Define $\mathcal{U}_1 = P(E_1 \oplus E_2) \setminus P(E_1)$ and
$\mathcal{U}_2 = P(E_1 \oplus E_2) \setminus P(E_2)$

A linear injection on fibres $g: E_1 \oplus E_2 \to \mathbb{RP}^\infty$ restricts to linear injections of E_1 and E_2. Thus the generator $x \in H^1(P(E_1 \oplus E_2), \mathbb{Z}_2)$ restricts to the generators defined for E_1 and E_2 in $H^1(P(E_1), \mathbb{Z}_2)$ and $H^1(P(E_2), \mathbb{Z}_2)$

Define
$$\omega_1 = \sum_i w_i(E_1)\, x^{m-i} \qquad m = \dim(E_1)$$
$$\omega_2 = \sum_i w_i(E_2)\, x^{n-i} \qquad n = \dim(E_2)$$

Then the cup product of ω_1 and ω_2 is
$$\omega_1\omega_2 = \sum_i \Big(\sum_{\ell+r=i} w_\ell(E_1)\, w_r(E_2) \Big) x^{m+n-i}$$

If $\omega_1\omega_2 = 0$, then this gives us a defining relation for $w(E_1 \oplus E_2)$. i.e. $w_i(E_1 \oplus E_2) = \sum_{\ell+r=i} w_\ell(E_1) \cup w_r(E_2)$

To show that $\omega_1\omega_2 = 0$, we know that ω_1 restricts to zero in $H^m(P(E_1), \mathbb{Z}_2)$. This implies that ω_1 pulls back to a class in the relative group $H^m(P(E_1 \oplus E_2), P(E_1))$ (coefficients in $\mathbb{Z}_2$ implied). $\mathcal{U}_2$ deformation retracts to $P(E_1)$, so this is equal to $H^m(P(E_1 \oplus E_2), \mathcal{U}_2)$. Then we have the following diagram:

open fact $P(E_1 \oplus E_2)$

$$\begin{array}{ccc} H^m(P(E_1 \oplus E_2), \mathcal{U}_2) \oplus H^n(P(E_1 \oplus E_2), \mathcal{U}_1) & \xrightarrow{\smile} & H^{m+n}(P(E_1 \oplus E_2), \mathcal{U}_1 \cup \mathcal{U}_2) \\ \downarrow & & \downarrow \end{array}$$

$$H^m(P(E_1 \oplus E_2)) \oplus H^n(P(E_1 \oplus E_2)) \xrightarrow{\smile} H^{m+n}(P(E_1 \oplus E_2))$$

So $\omega_1 \omega_2 = 0$.

(c) holds by definition of $w_i(E) = 0$ for $i > n$.

(d) see Hatcher. (Exercise?)

$\square$

Uniqueness for line bundles $L \to X$.

Recall the canonical bundle $E_1 = E_1(\mathbb{R}^\infty) \to G_1(\mathbb{R}^\infty) = \mathbb{R}P^1$

Property (c) implies that $w_i(E_1) = 0$ for $i > 0$

and (d) implies $w_1(E_1) = 1$ since it generates $H^1(\mathbb{R}P^\infty, \mathbb{Z}_2)$. Thus it is unique.

Suppose that θ is another characteristic class satisfying (a)–(d). Then, since

$$[X, \mathbb{R}P^\infty] \longrightarrow Vect_1(X)$$
$$f \longmapsto f^*(E_1)$$

Is bijective we have $L = f^*(E_1)$ for some f. Then

$$\theta_i(f^* E_1) = f^* \theta_i(E_1)$$
$$= f^* w_i(E_1) \quad \text{by uniqueness for } E_1$$
$$= w_i(f^* E_1)$$

$\therefore \quad \theta_i = w_i$ for line bundles

For general v.b.s, we can reduce to the case of line bundles by using the <u>Splitting principle</u>.

Examples:

1) if $E = X \times \mathbb{R}^n$ then $E = f^*(\{p\} \times \mathbb{R}^n)$. Thus $w_i(E) = 0 \ \forall i$. All trivial bundles have $w_i = 0 \ \forall i$.

2) Spheres: TS^n is stably trivial because $TS^n \oplus NS^n$

2) Spheres: TS^n is stably trivial because $TS^n \oplus NS^n$ is trivial, where

$$NS^n = \{ (x,v) \in S^n \times \mathbb{R}^{n+1} \mid x \parallel v \}$$

which is also trivial.

$$\therefore \quad w(TS^n) \smile w(NS^n) = w(S^n \times \mathbb{R}^{n+1})$$

$$\Rightarrow \quad w(TS^n) \smile 1 \qquad = \quad 1$$

$$\Rightarrow \quad w_i(TS^n) = 0 \qquad \text{for} \quad i > 0$$

Thus Stiefel-Whitney classes are not so useful fo spheres.

3) Canonical bundle over $\mathbb{R}P^\infty$ does not sum to a trivial bundle: Suppose there exist a bundle $F \to \mathbb{R}P^\infty$ such that $E_1 \oplus F$ is trivial. Then

$$w(E_1) \smile w(F) = 1$$

$$\Rightarrow \quad (1 + \omega) \smile w(F) = 1$$

$$\Rightarrow \quad (1 + \omega) \smile (1 + w_1(F) + w_2(F) + \cdots) = 1$$

$$\Rightarrow \quad 1 + (\omega + w_1(F)) + (w_2(F) + \omega w_1(F)) + \cdots$$

$$w_1(F) = \omega \ , \ w_2(F) = \omega^2 \ , \ w_3(F) = \omega^3$$

$$w(F) = 1 + \omega + \omega^2 + \omega^3 + \cdots$$

Contradicts $\qquad w_i(F) = 0 \quad \forall \ i > \dim(F)$

4) A bundle with $w_i \neq 0 \ \forall \ i \leq \dim(E)$:

Let $\pi_i : (\mathbb{R}P^\infty)^n \to \mathbb{R}P^\infty$ be the projection onto the ith copy of $\mathbb{R}P^\infty$. Then define $E = \pi_1^*(E_1) \oplus \cdots \oplus \pi_n^*(E_1)$

Then

$$\begin{aligned} w(E) &= \pi_1^*(1 + \omega_1) \smile \cdots \smile \pi_n^*(1 + \omega_n) \\ &= (1 + \eta_1) \smile \cdots \smile (1 + \eta_n) \\ &\in \mathbb{Z}_2[\eta_1, \cdots, \eta_n] \cong H^\bullet((\mathbb{R}P^\infty)^n, \mathbb{Z}_2) \end{aligned}$$

So $\quad w_i(E)$ is the symmetric polynomial on i-variables.