Today: The Thom Isomorphism.

References: Characteristic classes & K-Theory by O. Randal-Williams.

Aim: Reduce the study of $K^0(X)$ to an easier space.

Recall: $Y \subseteq X$ top. spaces.

The relative K-Theory $K^0(X,Y) := \tilde{K}^0(X/Y)$.

Problem: This doesn't make sense if X/Y is not compact Hausdorff.

Problem: Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a $\mathbb{C}$-v.b.

$E^{\#} := E \setminus s_0$ "the complement of the zero section"

Then $E/E^{\#}$ is not compact Hausdorff.

<u>Solution:</u> Study $\tilde{K}^0(Th(E))$ instead of $\tilde{K}^0(E/E^{\#})$.

<u>Def:</u> Let X be compact Hausdorff.

$\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ be a $\mathbb{C}$-v.b.

Define:

$$\mathbb{D}(E) := \{ v \in E \mid \langle v, v \rangle \leq 1 \}.$$

$$\mathbb{S}(E) := \{ v \in E \mid \langle v, v \rangle = 1 \}.$$

The

$$Th(E) := \mathbb{D}(E) / \mathbb{S}(E).$$

Why is this a good replacement:

<u>Claim:</u> (Thom isomorphism in cohomology)

Let R be any ring of coefficients.

$$H^i(E, E^{\#}, R) \cong \tilde{H}^i(Th(E)).$$

<u>Proof:</u>

Observe $r_E : E \longrightarrow \mathbb{D}(E)$

$r_{\#} : E^{\#} \longrightarrow \mathbb{S}(E).$

are deformation retracts.

We have a com. diag.

$$\begin{tikzcd}
\cdots & H^n(E^{\#}) \arrow[l] \arrow[d, "i_{\#}^*"] & H^n(E) \arrow[l] \arrow[d, "i_E^*"] & H^n(E, E^{\#}) \arrow[l] \arrow[d, "i^*"] & \arrow[l] \\
\cdots & H^n(\mathbb{S}(E)) \arrow[l] & H^n(\mathbb{D}(E)) \arrow[l] & H^n(\mathbb{D}(E), \mathbb{S}(E)) \arrow[l] & \arrow[l]
\end{tikzcd}$$

which is exact in the rows.

$i_{\#}^*$, i_E^* are isos.

$\therefore$ By the 5-lemma so is i^*

$\therefore$ $H^n(E, E^{\#}) \cong H^n(\mathbb{D}(E), \mathbb{S}(E)) \cong \tilde{H}^n(Th(E)).$

Another useful way of thinking of $Th(E)$.

<u>Claim:</u> If X is compact, then $Th(E)$ is homeomorphic to E^+, the 1-pt compactification of E.

<u>Proof</u> Let $\underline{\mathbb{C}}_X$ denote the trivial bundle on X.

We have two maps:

$$\begin{array}{ccccc} E & \xrightarrow{f} & \mathbb{P}(E \oplus \underline{\mathbb{C}}_X) & \hookleftarrow & \mathbb{P}(E). \\ v & \longmapsto & [v,1] \quad [v,0] & \longleftarrow\!\shortmid & [v] \end{array}$$

Then $\mathrm{im}(f) \cong \mathbb{P}(E \oplus \underline{\mathbb{C}}_X) \setminus \mathbb{P}(E)$.

$\Rightarrow \quad E^+ \cong \mathbb{P}(E \oplus \underline{\mathbb{C}}_X) / \mathbb{P}(E)$.

Choose some homeomorphism $\varphi : [0,1) \to [0,\infty)$
Scaling along radii in E, gives:

$$Th(E) \cong \mathbb{P}(E \oplus \underline{\mathbb{C}}_X)/\mathbb{P}(E) \cong E^+.$$

Claim: Let X be a compact Hausdorff space.
$\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ a $\mathbb{C}$-v.b. of rank n.

Then, $\exists \lambda_E \in \tilde{K}^0(Th(E))$ s.t.

① "The Thom isomorphism in K-Theory"

The map:

$$\Phi : K^0(X) \xrightarrow{\sim} K^0(\mathbb{D}(E)) \xrightarrow{\lambda_E \cdot} \tilde{K}^0(Th(E))$$

is an iso.

② $\forall f : X' \to X$, there's an induced map:
$Th(f) : Th(f^*(E)) \to Th(E)$, and

$$Th(f)^*(\lambda_E) = \lambda_{f^*(E)} \in K^0(Th(f^*E))$$

③ If $X = pt$, $\lambda_{\underline{\mathbb{C}}^n_{pt}} \in \tilde{K}^0(Th(\underline{\mathbb{C}}^n_{pt}))$
$= \tilde{K}^0(S^{2n})$
is a generator of the ring.

<u>Detour:</u> Exterior powers of vector bundles.

<u>Def:</u> For any $\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ a $\mathbb{C}$-v.b.

def $\Lambda^k(E) = \coprod_{x \in X} \Lambda^k(E_x)$.

endowed with a topology as used when constructing $E \otimes F$.

<u>Def:</u> $\Lambda_t(E) := \sum_{i=0}^{\infty} \Lambda^i(E)\, t^i \in K^0(X)[[t]]$.

<u>Note:</u>

- $\Lambda_t(E \oplus F) = \Lambda_t(E) \cdot \Lambda_t(F)$.

- $\Lambda_t(E - F) = \dfrac{\Lambda_t(E)}{\Lambda_t(F)}$

- $\Lambda_t : (K^0(X), +, 0) \rightarrow (K^0(X)[[t]]^\times, \times, 1)$.

Proof of claim:

We have an exact sequence:

$$\begin{tikzcd}
K^0(\mathbb{P}(E)) \arrow[d, "\partial"'] & K^0(\mathbb{P}(E \oplus \underline{\mathbb{C}}_X)) \arrow[l, two heads, "i^*"'] & \tilde{K}^0(Th(E)) \arrow[l, "q^*"'] \\
\tilde{K}^{-1}(Th(E)) \arrow[r, "q^*"] & K^{-1}(\mathbb{P}(E \oplus \underline{\mathbb{C}}_X)) \arrow[r, two heads, "i^*"] & K^{-1}(\mathbb{P}(E)) . \arrow[u, "\partial"']
\end{tikzcd}$$

The maps i^* are surjective [Projective bundle formula].

If we write ω_F for the canonical line bundle on $\mathbb{P}(F)$,

Then $\omega_E = i^*(\omega_{E \oplus \underline{\mathbb{C}}_X})$. So we abuse notation and ω for both.

$\tilde{K}^0(Th(E))$

$$\cong \ker\left[\frac{K^0(X)[\omega]}{\left(\sum_{i=0}^{n+1}(-1)^i \Lambda^i(E\oplus \underline{\mathbb{C}}_X)\ \omega^{n+1-i}\right)} \xrightarrow{i^*} \frac{K^0(X)[\omega]}{\sum_{i=0}^{n}(-1)^i\Lambda^i(E)\,\omega^i}\right].$$

$$= \ker\left[\frac{K^0(X)[\omega]}{\omega^{n+1}\Lambda_{-\omega^{-1}}(E)\cdot(1-\omega^{-1})} \longrightarrow \frac{K^0(X)[\omega]}{\omega^n\cdot\Lambda_{-\omega^{-1}}(E)}\right].$$

Note that ω^n is a unit.

$$\Rightarrow \Lambda_{-\omega^{-1}}(E) \in \ker(i^*)$$

$$\Rightarrow \overline{\Lambda_{-\omega^{-1}}(E)} \in \ker(i^*)$$

$$\Rightarrow \Lambda_{-\omega}(\overline{E}) \in \ker(i^*).$$

$$\therefore \Lambda_{-\omega}(\overline{E}) \in \tilde{K}^0(Th(\overline{E})).$$

Define λ_E to be the unique object such that $q^*(\lambda_E) = \Lambda_{-\omega}(\overline{E})$.

To prove (2): Note it is natural by construction.

To prove (3):

If $X = pt.$, $\Lambda^i(\underline{\mathbb{C}}^n_X) = \binom{n}{i} \in K^0(pt)$.

$$\Rightarrow \Lambda_{-\omega}(\underline{\mathbb{C}}^n_X) = (1-\omega)^n$$

$\therefore$ this generates $K^0(S^{2n})$.

To prove (1):

$$\Phi : K^0(X) \xrightarrow{\sim} \frac{K^0(X)[\omega]}{(1-\omega)} \underset{\sim}{\xrightarrow{\lambda_E \cdot}} \frac{(\lambda_E)}{(\lambda_E \cdot (1-\omega))} = \ker i^*$$

$$\cong \tilde{K}^0(Th(E))$$

$\therefore$

The zero section $s_0 : X \longrightarrow E$

extends to a based map: $s_0 : \underset{\substack{\| \\ X \sqcup \{\text{base pt}\}}}{X_+} \longrightarrow E^+$.

which induces a morphism:

$$s_0^* : \tilde{K}^0(Th(E)) \longrightarrow K^0(X_+) \cong K^0(X).$$

<u>Def:</u> "K-Theory Euler class".

For any $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ a $\mathbb{C}$-v.b. set:

$$e^K(E) := s_0^*(\lambda_E) \in K^0(X).$$

<u>Claim:</u> $c_i^K(E) = \Lambda^i(E) \in K^0(X)$. "ith Chern class".

$$e^K(E) = \Lambda_{-1}(\overline{E}) \in K^0(X)$$

Claim: "K-Theory Gysin Sequence"

We have a sequence

$$\begin{tikzcd}
K^0(S(E)) \arrow[d, "p_!"] & K^0(X) \arrow[l, "p^*"'] & K^0(X) \arrow[l, "e^K(E)\cdot"'] \\
K^{-1}(X) \arrow[r, "e^K(E)\cdot"] & K^0(X) \arrow[r, "p^*"] & K^{-1}(S(E)) \arrow[u, "p_!"']
\end{tikzcd},$$

where $p : S(E) \to X$.

$p_!$:

$$K^i(S(E)) \longrightarrow \tilde{K}^{i+1}(D(E)/S(E))$$
$$\cong$$
$$\tilde{K}^{i+1}(Th(E)) \xrightarrow{\Phi^{-1}} K^{i+1}(X)$$