Def: Let X be a topological space, $A \subset X$ a subspace.

We define

$\pi_n(X,A) :=$ htpy classes of maps

$$\begin{tikzcd} D^n \arrow[r] & X \\ \partial D^n \arrow[u, hook] \arrow[r] & A \arrow[u, hook] \end{tikzcd}$$

where homotopy is taken to keep ∂D^n in A.

CW complexes are designed to be "totally determined" by homotopy groups.

That is, CW complexes are spaces built out of disks in the following way:

$$\begin{tikzcd} \coprod_{n \text{ cells}} D_i^n \arrow[r] & X_n \\ \coprod_{\substack{n \text{ cells} \\ D_i}} \partial D_i^n \arrow[u, hook, "\iota"] \arrow[r, "\sqcup \phi_i"'] & X_{n-1} \arrow[u, hook] \end{tikzcd}$$

ϕ_i are our attaching maps.

is a pushout of topological spaces.

So lets try construct a map

$$X \xrightarrow{f} Y$$ where X is a CW complex.

Existence:

$$\begin{tikzcd}
\bigsqcup D_i^n \arrow[r] \arrow[rr, bend left] & X_n \arrow[r, dashed, "\exists \tilde{f}"] & Y \\
\bigsqcup \partial D_i^n \arrow[r] \arrow[u, hook] & X_{n-1} \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

So $\exists \hat{f}$ lifting f iff all

$\partial D_i^n \xrightarrow{\phi_i} X_{n-1} \longrightarrow Y$ are 0 in

$\pi_{n-1}(Y)$.

Uniqueness:

$$\begin{tikzcd}
D^n \arrow[r] \arrow[rr, bend left, "\tilde{f}_2"] & S^n \arrow[r, dashed] & Y \\
\partial D^n \arrow[r, hook] \arrow[u, hook] & D^n \arrow[u] \arrow[ur, "\hat{f}_1"'] &
\end{tikzcd}$$

homotopic iff $S^n \longrightarrow Y$ is 0 in $\pi_n(Y)$.

Fibre bundles:

$$\begin{tikzcd}
F \arrow[r] & E \arrow[d] \\
& X
\end{tikzcd}$$

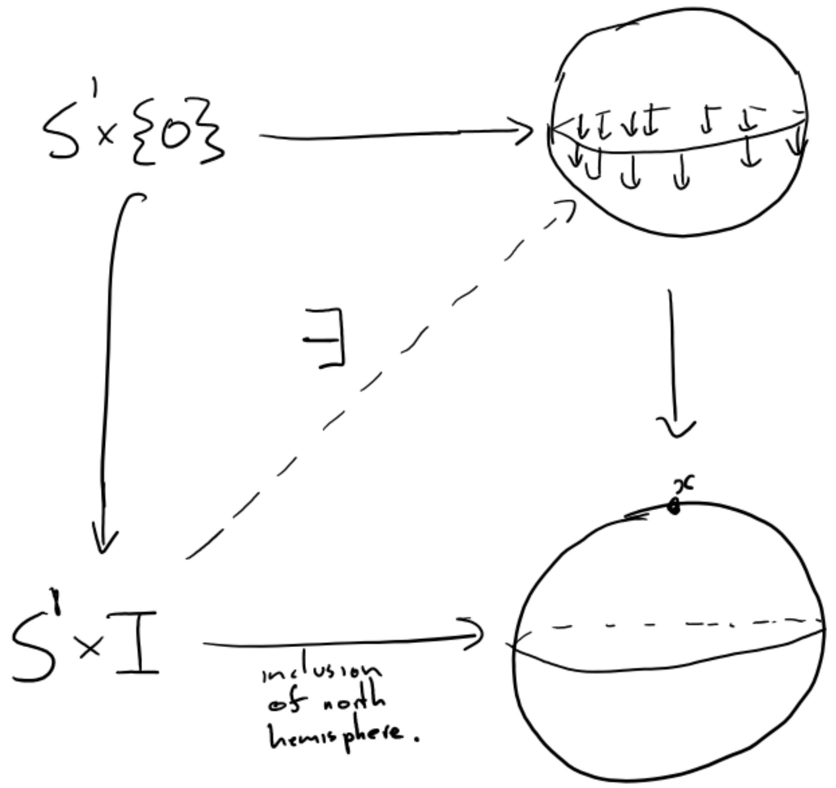
homotopy lifting property:

$$\begin{tikzcd} Y \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d] \\ Y \times I \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$$

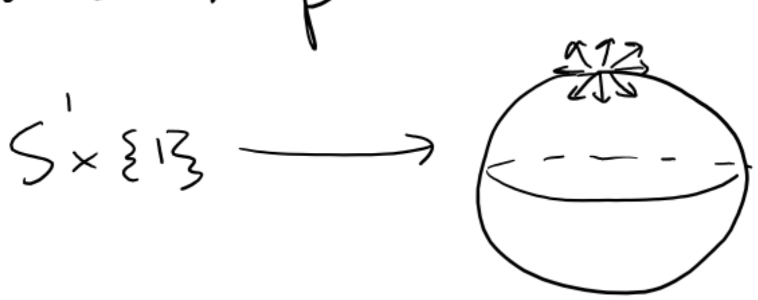
holds for fibre bundles.

So we can <u>lift</u> our homotopies.

Eg: Look at the unit S^1 bundle E over S^2, tangent vectors of norm 1.



homotopy lift moves the tangent vectors to the north pole, ending with a map



So the nullhomotopy lifts to a map $S^1 \longrightarrow F_x$ fibre over north pole.

General setup:

$\boxed{\text{Assumptions:}}$

X connected CW complex, A sub-CW complex, F path connected,

$\pi_1(F) \curvearrowright \pi_n(F)$ trivial.

$\pi_1(X) \curvearrowright \pi_n(F)$ trivial also.

$\boxed{\text{To avoid local systems}}$

Want a section of
$$\begin{array}{c} E \\ \downarrow \\ X \end{array}$$

So let $s: X^n \longrightarrow E$ be a section over n-skeleton.

We define a map

$$\omega_s : \pi_{n+1}(X^{n+1}, X^n) \longrightarrow \pi_n(F)$$

as follows:

$$
\begin{tikzcd}
\partial D^{n+1}\times\{1\} \arrow[rr, "\exists"] \arrow[d, hook] & & F_x \arrow[r, hook] & E \arrow[d] \\
\partial D^{n+1}\times I \arrow[r] \arrow[urrr, "\exists"] & D^{n+1} \arrow[r] & X^{n+1} \arrow[r, hook] & X \\
& \partial D^{n+1}\times\{0\} \arrow[ul, hook] \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\alpha"] & X^n \arrow[u, hook] \arrow[ur, hook] \arrow[uur, dashed, "s"] &
\end{tikzcd}
$$

That is, use the section s over X^n to lift the null homotopy in X^{n+1} to a homotopy in E, which maps $\partial D^{n+1}\times\{1\}$ to the fibre over x vertex of the null homotopy.

So we get a map (based on s)

$$\omega_s : \pi_{n+1}(X^{n+1}, X^n) \longrightarrow \pi_n(F).$$

Using the attaching maps of $n+1$ cells, we get

$$\omega_s : H_{n+1}(X^{n+1}, X^n) \longrightarrow \pi_n(F)$$

Lemma 3.18 Hatcher:

$\boxed{\text{This map } \omega_s : H_{n+1}(X^{n+1}, X^n) \longrightarrow \pi_n(F) \text{ is a cellular cocycle, hence an element } [\omega_s] \text{ of } H^{n+1}(X, \pi_n(F)).}$

Note if s is defined over A, then this cocycle vanishes in $H^{n+1}(X, \pi_n(F))$, so gives an ele of $H^{n+1}(X, A, \pi_n(F))$.

<u>Prop</u>: As s varies over sections $X^n \cup A \longrightarrow E$ extending the given section over $X^{n-1} \cup A$, the ω_s vary over coboundaries, so

$[\omega_s] \in H^{n+1}(X, A, \pi_n(F))$

is zero <u>iff</u> $s|_{X^{n-1} \cup A}$ extends to a section over $X^{n+1} \cup A$.

Upshot:

$\boxed{\text{Extending a section } s \text{ from } X^n \cup A \text{ to } X^{n+1} \cup A \text{ has a well defined \underline{obstruction} in } H^{n+1}(X, A, \pi_n(F)) \text{ depending only on } s|_{X^{n-1} \cup A}.}$

A section of
$$\begin{array}{c} E \times I \\ \downarrow \\ X \times I \end{array}$$
is a homotopy of sections of $\begin{array}{c} E \\ \downarrow \\ X \end{array}$,

So we want to consider sections over $X \times I$ extending a given section on
$$\underbrace{X \times \partial I}_{\substack{\text{two endpoints} \\ s_0, s_1}} \cup \underbrace{A \times I}_{\substack{\text{want our homotopy} \\ \text{to be fixed on } A.}}.$$

So the obstruction to finding a homotopy (fixed on A) between $s_0, s_1 : X^n \longrightarrow E$ which agree on A is an element of

$$H^{n+1}(X \times I, X \times \partial I \cup A \times I, \pi_n(F))$$

$$\cong$$

$$\tilde{H}^{n+1}\left(\frac{X \times I}{X \times \partial I \cup A \times I}, \pi_n(F)\right)$$

$$\cong$$

$$\tilde{H}^{n+1}(\Sigma X / \Sigma A, \pi_n(F))$$

$$\cong$$

$$\tilde{H}^{n+1}(\Sigma X, \Sigma A, \pi_n(F))$$

$$\cong$$

$$H^n(X, A, \pi_n(F))$$

So given s a section on $A \subset X$, the obstructions to existence and uniqueness over X^i from X^{i-1} lie in

$$\underbrace{H^i(X, A, \pi_{i-1}(F))}_{\text{existence}} \quad \text{and} \quad \underbrace{H^i(X, A, \pi_i(F))}_{\text{uniqueness}}$$

So now assume the fibre F is $(n-1)$ connected, then all these vanish up to a canonical element of $H^{n+1}(X, A, \pi_n(F))$.

That is, we have a section s over $X^n \cup A$, and its restriction to $X^{n-1} \cup A$ is unique up to homotopy.

This is the <u>primary obstruction.</u>

Examples:

1. If E is a real vector bundle over X, of dimension n, then its associated sphere bundle has fibre S^{n-1}, so we have a primary obstruction living in
$$H^n(X, \pi_{n-1}(S^{n-1})) \cong H^n(X, \mathbb{Z}).$$

This is the euler class, up to $\pm$ / orientation issues.

2. E real of deg n, have associated k-frame bundle with fibre the Steifel manifold $V_k(\mathbb{R}^n)$.

Fact: The first nonzero homotopy group of $V_k(\mathbb{R}^n)$ is π_{n-k}, and is either $\mathbb{Z}$ or $\mathbb{Z}/2$.

So reducing coefficients mod 2, we get the image of the primary obstruction in

$$H^{n-k+1}(X, \mathbb{Z}/2).$$

This is the $(n-k+1)$th Steifel whitney class.