**Second talk**. Perverse sheaf examples

**Reference:** Juteau – Williamson – Mautner

Let $\mathcal{F}$ be perverse on $X = \bigcup_{s \in S} X_s$

**Then:**

(i) $\mathcal{F}|_{X_s}$ is a local system on $X_s$

(ii) For any point $x_s \in X_s$ $\quad (\mathcal{F}|_{X_s})_{x_s}$ is constant on $x_s \in X_s$

We define the table of stalks for $\mathcal{F}$ by

**Def**. Table of stalks

$X = \bigsqcup_{s \in S} X_s$ is a strat.

Some local system since $\mathcal{F}$ is perverse

| Strata | $d-1$ | $d$ | $d+1$ |
|---|---|---|---|
| $X_{s_n}$ | | $H^d(\mathcal{F})\big|_{X_{s_n}}$ | |
| $X_{s_{n-1}}$ | | | |
| $\vdots$ | | | |
| $X_{s_0}$ | | | |

If $\mathcal{F} \in P_X$ then $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ both have tables that look

perverse sheaves

$\dim(X_{s_i}) \leq \dim(X_{s_{i+1}})$

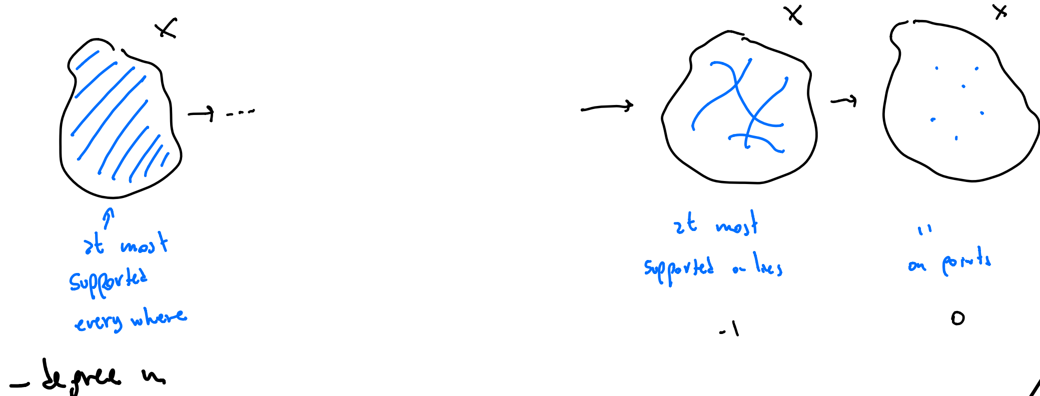
So the table for perverse sheaves looks like

| | $-\dim X_n = -\dim X$ | $\cdots$ | | $-\dim X_2$ | $-\dim X_1$ | $-\dim X_0$ |
|---|---|---|---|---|---|---|
| $X_n$ | * | $\cdots$ | 0 | 0 | 0 | 0 |
| $\vdots$ | * | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $X_2$ | * | $\cdots$ | * | 0 | $\vdots$ | $\vdots$ |
| $X_1$ | * | | * | * | 0 | 0 |
| $X_0$ | * | $\cdots$ | * | * | * | * |

$IC(X_0, \mathcal{L})$

Another picture of perverse sheaves

Support of $\mathcal{F}$ in blue




$IC(X_0, \mathcal{L})$ : irreducible local system

## Deligne's construction

$X$ = top. space.

$S_m$ = union of all strata of dim $m$.

$X_m$ = union of all strata of dim $\geq m$

We have inclusions:

$$X_{\dim X} \xhookrightarrow{j_{\dim X-1}} X_{\dim X-1} \xhookrightarrow{j_{\dim X-2}} \cdots \hookrightarrow X_1 \hookrightarrow X_0 = X$$

Thm. For an irreducible local system $\mathcal{L}$ on $s \in S$

$$IC(S,\mathcal{L}) = (\tau_{\leq -1} \circ j_{0*}) \circ \cdots \circ (\underbrace{\tau_{\leq -d_s}}_{\text{truncation w.r.t standard } t\text{-structure}} \circ j_{d_s-1*}) \underbrace{\mathcal{L}[d_s]}_{X_{d_s}}$$

$d_s = \dim s$.

$D^{\leq 0}$ = complexes whose cohomology vanishes for $i > 0$.

Example. Nilpotent cone in $sl_2$.

$$\mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \middle| \; x^2 + yz = 0 \right\} \subset \mathbb{C}^3$$

There is a homeo:

$$\mathbb{C}^2 / (u,v) = (-u,-v) \xrightarrow{\sim} \mathcal{N}$$

$$(u,v) \longmapsto \begin{pmatrix} uv & v^2 \\ u^2 & -uv \end{pmatrix}$$

Consider $P_{\mathcal{N}}$ = perverse sheaves w.r.t. strat by $SL_2$-orbits.

$SL_2 \circlearrowright sl_2$

group

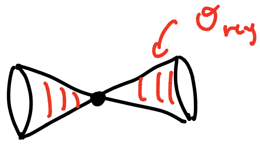
There are 2 nilpotent JNF:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The strata:

$$S = \{ \{0\},\ \mathcal{O}_{reg} = \mathcal{N} \setminus \{0\} \}$$

A real picture:




$$\mathcal{O}_{reg} = \mathcal{N} \setminus \{0\} = \mathbb{C}^2 \setminus \{0\} \Big/ (u,v) = -(u,v) \overset{\text{homotopy equivalence}}{\simeq} S^3 \big/ x = -x \simeq \mathbb{RP}^3.$$

What are the IC sheaves on $\mathcal{N}$?

$IC(\{0\}, \underline{\mathbb{k}}_{\{0\}}) = \underline{\mathbb{k}}_{\{0\}}$ ← skyscraper sheaf

What is $IC(\mathcal{O}_{reg}, \underline{\mathbb{k}}_{\mathcal{O}_{reg}}[2])$?

Table for $j_* \underline{\mathbb{k}}_{\mathcal{O}_{reg}}[2]$

$$\boxed{j : \mathcal{O}_{reg} \hookrightarrow \mathcal{N}}$$

| | $-2$ | $-1$ | $0$ | $1$ |
|---|---|---|---|---|
| $\mathcal{O}_{reg}$ | $\mathbb{k}$ | $0$ | $0$ | $0$ |
| $\{0\}$ | $\mathbb{k}$ | $(\mathbb{k})_2$ | $(\mathbb{k})_2$ | $\mathbb{k}$ |

$j^* j_* \simeq id$

Stalks for the IC sheaf.

depends on $\mathbb{k}$.

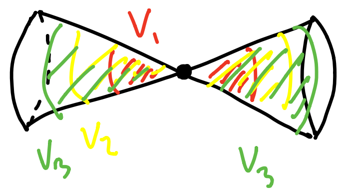
$\mathbb{k}$ can be any field.

what is $j_* \underline{\mathbb{k}}_{\mathcal{O}_{reg}}[2]$?

$$(\mathbb{k})_2 = \begin{cases} \mathbb{k} & \text{if char } \mathbb{k} = 2 \\ 0 & \text{o/w} \end{cases}$$

$$\underbrace{(j_* \underline{\mathbb{k}}_{\mathcal{O}_{reg}})_0}_{\text{stalk at } 0} \simeq \lim_{V \ni 0} R\Gamma(\underbrace{V \setminus \{0\}}_{j^{-1}(V)}, \mathbb{k})$$

↖ lets take a base of these sets to compute this




$V_n$ = balls of radius $n$ intersect $\mathcal{N}$.

$$R\Gamma(V_n \setminus \{0\}, \mathbb{k}) = R\Gamma(\mathcal{O}_{reg}, \mathbb{k})$$

$$(j_* \underline{\mathbb{k}}_{\mathcal{O}_{reg}})_0 \simeq R\Gamma(\mathcal{O}_{reg}, \mathbb{k}) = R\Gamma(\mathbb{R}\mathbb{P}^3, \mathbb{k})$$

$$\simeq \mathbb{H}(\mathbb{k} \xrightarrow{0} \mathbb{k} \xrightarrow{2} \mathbb{k} \xrightarrow{0} \mathbb{k})$$

$$= \begin{cases} \mathbb{k} \quad 0 \quad 0 \quad \mathbb{k} & \text{char} \neq 2 \\ \mathbb{k} \quad \mathbb{k} \quad \mathbb{k} \quad \mathbb{k} & \text{char} = 2 \end{cases}$$