Some motivation / "big picture stuff": the p-adic story.

· p-adic numbers

$$\mathbb{Z}_p = \left\{ \sum_{n=0}^{\infty} a_n p^n \;\middle|\; a_n \in \{0, 1, \dots, p-1\} \right\}$$

$\mathbb{Z}_p$ has no zero divisors $\rightsquigarrow \mathbb{Q}_p$.

"One of the most open problems in rep-theory"

$G(\mathbb{Q}_p)$ is a p-adic group

Goal: Classify all irreps of $G(\mathbb{Q}_p)$.

Start with 'nice' groups: semisimple, reductive, split

and 'nice' irreps: smooth, admissible,

↳ are either 1-dim'l or ∞-dim'l

We get this link:

$$\left\{ \begin{matrix} \infty\text{-dim'l rep}^{\text{n}}\text{s} \\ \text{of } G(\mathbb{Q}_p) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{f.d. rep}^{\text{n}}\text{s of the} \\ \text{affine Hecke algebra} \end{matrix} \right\}$$

$\mathbb{C}[I \backslash G(\mathbb{Q}_p)/I]$ "Langlands duality"

Classification of irreps of $\mathbb{H}$:

Recall for semisimple $\mathfrak{g}$ over $\mathbb{C}$:

(1) Choose 'central character' (weight) of $(\mathfrak{g}, \mathfrak{h})$

(2) Construct 'Verma modules' V_χ

(3) 'Locate' irreps V_χ $\rightsquigarrow$ classification

In the case of the affine Hecke algebra:

(1) Choose a 'central character' $a = (s, t) \in G \times \mathbb{C}^\times$
and specialise $\mathbb{H}$ to $\mathbb{H}_a$.

(2) Construct "standard modules"
via equivariant K-theory & Borel-Moore homology.

(3) 'Locate' irreps via perverse sheaves.

The finite case

Let
- $T \subset G$ maximal torus
- $R(T)$ be its representation ring:
  $R(T) \cong (\langle [V] : V \text{ a f.d. } T\text{-irrep} \rangle, \oplus, \otimes)$
- $P = \mathrm{Hom}(T, \mathbb{C}^\times)$ weight lattice
  $P \cong \langle \lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}^\ell \rangle$ (rank of $T = \ell$).
- $P^\vee = \mathrm{Hom}(\mathbb{C}^\times, T)$ coweight lattice
- $R \subset P$ root system
- $R^\vee \subset P^\vee$ coroot system, $R \longleftrightarrow R^\vee$.
- $S \subset R$ choice of simple roots.

Weyl group: $W = \left\langle s \in S \;\middle|\; \begin{array}{l} s_\alpha^2 = 1 \\ s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots \quad m(\alpha,\beta) \text{ factors} \end{array} \right\rangle$

$W \circlearrowright P \rightsquigarrow \hat{W} \text{ or } W_{aff} = W \ltimes P.$ affine weyl group.

The **Hecke algebra** $\mathcal{H}_W$ is a $\mathbb{Z}[q,q^{-1}]$-algebra

$$\left\langle T_s \;,\; s\in S \;\middle|\; \begin{array}{l} \cdot\, (T_{s_\alpha}-1)(T_{s_\alpha}+q) = 0 \\ \cdot\, T_{s_\alpha} T_{s_\beta} T_{s_\alpha} \cdots = T_{s_\beta} T_{s_\alpha} T_{s_\beta} \cdots \quad m(\alpha,\beta) \text{ factors} \end{array} \right\rangle$$

Fact: There is "standard basis" $\{T_w \,,\, w\in W\}$, w/ different relations.

The **affine Hecke algebra** $\mathbb{H}$ is a $\mathbb{Z}[q,q^{-1}]$-algebra with basis $\{ e^\lambda \cdot T_w \mid \lambda\in P,\, w\in W\}$ such that

1) $\operatorname{span}\{T_w\} \cong \mathcal{H}_W$
2) $\operatorname{span}\{e^\lambda\} \cong R(T)[q,q^{-1}]$
3) some other formulae...

Fact: $$\begin{aligned} Z(\mathbb{H}) &\cong R(T)^W [q,q^{-1}] \\ &\cong R(G)[q,q^{-1}] \\ &\cong R(G\times\mathbb{C}^\times) \end{aligned}$$

all rep^{n}s in $R(G\times\mathbb{C}^\times)$ $\updownarrow$ semisimples (s,t).

Specifically, $a=(s,t)\in G\times\mathbb{C}^\times$ semisimple

$\rightsquigarrow$ 1-dim rep^n of $Z(\mathbb{H}) \cong R(G\times\mathbb{C}^\times)$ $\ni [V]$ $\quad \chi_V : G\times\mathbb{C}^\times \to \mathbb{C}$, $(s,t)\mapsto \operatorname{tr}(\ldots)$

$$\begin{aligned} \mathrm{eval}_a : R(G\times\mathbb{C}^\times)\times\mathbb{C} &\longrightarrow \mathbb{C} \\ ([V], z) &\longmapsto \chi_V(a)\cdot z. \end{aligned}$$

denote this module by $\mathbb{C}_a$.

Then specialised affine Hecke algebra is

$$\mathbb{H}_a := \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H}.$$

Springer theory

G a $\mathbb{C}$ semisimple simply connected $\underbrace{\quad}_{}$ group. reductive

$\mathfrak{g}$ its Lie algebra

$\mathcal{B}$ its flag variety $= \{$ Borel subalgebras of $\mathfrak{g} \}$ $\circlearrowleft \mathbb{C}^\times$ trivial.

$\mathcal{N}$ its nilpotent cone $= \{ x \in \mathfrak{g} \ : \ ad\, x$ is nilpotent $\}$ $\circlearrowleft z^{-1}$.

$\tilde{\mathcal{N}}$ the Springer resolution

$$\tilde{\mathcal{N}} = \{ (x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \ | \ x \in \mathfrak{b} \}.$$

$$\mu : \tilde{\mathcal{N}} \longrightarrow \mathcal{N}$$
$$(x, \mathfrak{b}) \longmapsto x$$

Z the Steinberg variety $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ $\circlearrowleft \mathbb{C}^\times$

$$\cong \{ (x, \mathfrak{b}, \mathfrak{b}') \ | \ x \in \mathfrak{b} \cap \mathfrak{b}' \}.$$

$G \times \mathbb{C}^\times \circlearrowright Z : (x, \mathfrak{b}, \mathfrak{b}') \longmapsto (z^{-1} \cdot g x g^{-1}, g \mathfrak{b} g^{-1}, g \mathfrak{b}' g^{-1}).$

Classifying irreps of $\mathbb{H}$

(lots of technical ideas skipped!)

① **Schur's Lemma:** $Z(\mathbb{H})$ acts on M by scalar mult.

$\left\{ \begin{array}{l} \text{For any irreducible } \mathbb{H}\text{-modules} \\ \exists\ a = (s, t) \in G \times \mathbb{C}^\times \text{ semisimple such that} \\ \text{action } \mathbb{H} \circlearrowright M \text{ factors through some } \mathbb{H}_a\text{-action.} \end{array} \right\}$

Moral: To study $\mathbb{H}$, we look at $\mathbb{H}_a$.

② Convolution in Borel-Moore Homology

$$\mathbb{H}_a \cong H_*^{BM}(Z^a)$$

↑ proven via equivariant K-theory.

③ Define standard modules & locate irreps.

Work with $G = SL_n\mathbb{C}$ for simplicity $\quad a = (s,t)$

Let $\mathcal{N}^a = \{ x \in \mathcal{N} : ad_s(x) = t \cdot x \}$

Let $\mathcal{B}_x^s = \{$ Borel subalgebras fixed by s and $x \in \mathcal{N}^a \}$.

Standard $H_\bullet(Z^a)$-modules

$$K_{a,x} := H_\bullet^{BM}(\mathcal{B}_x^s).$$

Associate these new modules $L_{a,x}$

via "smooth tubular nbhd" — skip.

<u>Classification thm of irreps (Deligne - Langlands - Lusztig)</u>:

For $a = (s,t) \in G \times \mathbb{C}^\times$ semisimple, $t \in \mathbb{C}^\times$ is <u>not</u> a root of unity,

the collection

$$\{ L_{a,x} \}, \quad (a,x) \in (G \times \mathbb{C}^\times) \times \mathcal{N}^a / G\text{-conjugacy classes},$$

is a complete collection of $\mathbb{H}_a$-irreps.

Part of the proof of this conjecture involves perverse sheaves, and the fact that the $H_\bullet(Z^a)$-irreps are essentially $H_\bullet(Z)$-irreps.

Perverse sheaves in the proof:

$D^b(X)$ bounded derived category of constructible complexes on X.

Define sheaf Ext in $D^b(X)$ to be

$$\mathrm{Ext}^k_{D^b(X)}(A^\cdot, B^\cdot) := \mathrm{Hom}_{D^b(X)}(A^\cdot, B^\cdot[k]).$$

In the Springer resolution $\mu: \hat{\mathcal{N}} \longrightarrow \mathcal{N}$.

$\mathcal{N}$ has stratification $\mathcal{N} = \sqcup \mathbb{O}$. ($\mathbb{O}$ is a G-orbit).

$\hat{\mathcal{N}}$ has stratification $\tilde{\mathcal{N}} = \sqcup \hat{\mathbb{O}}$, $\hat{\mathbb{O}} = \mu^{-1}(\mathbb{O})$

We define the constant perverse sheaf $\mathcal{C}_{\tilde{\mathcal{N}}}$ by

$$\mathcal{C}_{\tilde{\mathcal{N}}}|_{\tilde{\mathcal{N}}_\alpha} = \mathbb{C}_{\tilde{\mathcal{N}}_\alpha}[\dim_{\mathbb{C}} \tilde{\mathcal{N}}_\alpha].$$

Theorem: $H_\bullet(Z) \cong \mathrm{Ext}^\bullet_{D^b(\mathcal{N})}(\mu_* \mathcal{C}_{\tilde{\mathcal{N}}}, \mu_* \mathcal{C}_{\tilde{\mathcal{N}}})$.

<u>not</u> as graded algebras

Theorem: (Equivariant decomposition (BBD), Bernstein–Lunts)

$$\mu_* \mathcal{C}_{\tilde{\mathcal{N}}} = \bigoplus_{\substack{i \in \mathbb{Z} \\ \phi = (\mathbb{O}, \chi)}} L_\phi(i) \otimes IC_\phi[i].$$

$\phi = (\mathbb{O}, \chi)$: $\mathbb{O}$ is a G-orbit in $\mathcal{N}$, χ is irred. G-equivariant loc. system

$IC_\phi = IC(\mathbb{O}, \chi)$ the associated int. coh. complex

$L_\phi(i)$ are "certain f.d. vec. spaces" ← given in the proof in BBD.

not necessarily nonzero.

$$H_\bullet(Z) \cong \mathrm{Ext}^\bullet_{D^b(\mathcal{N})}(\mu_* \mathcal{C}_{\tilde{\mathcal{N}}}, \mu_* \mathcal{C}_{\tilde{\mathcal{N}}})$$

$$\cong \bigoplus_{\substack{i,j,k \in \mathbb{Z} \\ \phi, \psi}} \mathrm{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \mathrm{Ext}^k_{D^b(\mathcal{N})}(IC_\phi[i], IC_\psi[j]).$$

$$\cong \quad \vdots \qquad \text{(properties of sheaf Ext)}$$

$$\cong \underbrace{\Big(\bigoplus_{\phi} \underbrace{\mathrm{End}\, L_\phi}_{\text{matrix algebra}}\Big)}_{\substack{\text{semisimple} \\ \text{algebra}}} \oplus \underbrace{\Big(\bigoplus_{\substack{k>0 \\ \phi, \psi}} \mathrm{Hom}_{\mathbb{C}}(L_\phi, L_\psi) \otimes \mathrm{Ext}^k_{D^b(\mathcal{N})}(IC_\phi, IC_\psi)\Big)}_{\text{concentrated in deg} > 0.}$$

<u>Fact</u>: is a nilpotent ideal.

<u>Fact</u>: is the radical.

Projection $H_\bullet(Z) \twoheadrightarrow \mathrm{End}\, L_\phi$ gives all irreps of $H_\bullet(Z)$ (assuming $L_\phi \neq 0$).

$\rightsquigarrow$ gives all irreps of $H_\bullet(Z^a)$

$\therefore$ of H_a.