k: field

G: connected reductive complex alg. group.

W: Weyl group

$\mathcal{N}$: nilpotent cone

The springer correspondence is an injective map

$$\nu: \operatorname{Irr}(k[W]) \hookrightarrow \operatorname{Irr}(\operatorname{Perv}_G(\mathcal{N}, k))$$

goal for today: understand ν

$\Rightarrow$ a geometric description of $k[W]$ and a classification of its irreducibles over W.

$x \in \mathfrak{g}$ is nilpotent if it acts by a nilpotent operator on every f.d. representation of G.

Example $G < Gl_n$ $\mathfrak{g} < \mathfrak{gl}_n$

$A \in \mathfrak{g}$ is nilpotent iff A nilpotent matrix

The nilpotent cone is

$$\mathcal{N} = \mathcal{N}_G := \{\text{nilpotent elements in } \mathfrak{g}\} \subset \mathfrak{g}$$

Example $G = SL_2$

$$\mathcal{N}_G = \{ A \in \mathfrak{sl}_2 \mid \operatorname{rank} A \leq 1 \}$$

singular point at 0

$$
\begin{array}{l}
\mathfrak{g} \circlearrowleft G \\
\cup \qquad \text{(closed subvariety} \\
\mathcal{N} \circlearrowleft G
\end{array}
$$

today we study G-equivariant perverse sheaves on $\mathcal{N}$

Facts on $\mathcal{N}$

① $G \to G/Z(G)$ induces $\mathcal{N}_G \xrightarrow{\cong} \mathcal{N}_{G/Z(G)}$

② $\mathcal{N}$ is irreducible

③ $\mathcal{N}$ has finitely many G-orbits

$\Rightarrow$ $\mathcal{N}$ is stratified by its G-orbits

THE SITUATION

$$T \subset B \subset G$$

maximal torus   Borel

$$\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$$

$$\begin{array}{c} \cup \\ \mathfrak{n} = [\mathfrak{b},\mathfrak{b}] \\ \cap \\ \mathcal{N} \end{array}$$

nil radical

$$W := N_G(T)/T$$

Weyl group

$$\mathcal{B} := G/B$$

flag variety

THE SPRINGER RESOLUTION

Let $\tilde{\mathcal{N}} := G \overset{B}{\times} \mathfrak{u} = \boxed{G \times \mathfrak{u} / B}$

$\big(h \cdot (g, x) = (g h^{-1}, \operatorname{ad}(h) \cdot x) \big)$

The springer resolution is the map

$$\mu : \tilde{\mathcal{N}} \longrightarrow \mathcal{N} : (g, x) \longmapsto \operatorname{ad}(g) \cdot x$$

The Springer fiber over $x \in \mathcal{N}$ is

$$\mathcal{B}_x = \mu^{-1}(x) \subset \tilde{\mathcal{N}}$$

Remark

$$\begin{tikzcd}
\tilde{\mathcal{N}} = G \overset{B}{\times} \mathfrak{u} \arrow[rr, "\sim"] \arrow[dr, "\mu"'] & & \{ (gB, y) \in \mathcal{B} \times \mathcal{N} \mid y \in \operatorname{ad}(g)\mathfrak{u} \} \arrow[dl, "\mathrm{pr}_2"] \\
& \mathcal{N} &
\end{tikzcd}$$

$$(g, x) \longmapsto (gB, \operatorname{ad}(g)x)$$

Hence:

① $\mathcal{B}_x = \mu^{-1}(x) = \{ gB \mid x \in \mathrm{ad}(g)\,\mathfrak{u} \} \subset \mathcal{B}$

② $\tilde{\mathcal{N}} \xrightarrow{\ pr_1\ } \mathcal{B}$ (good char)

is a vector bundle over $\mathcal{B}$

(with fibers $pr^{-1}(b) = \mathfrak{u}$)

cotangent bundle

In fact:

$$\begin{tikzcd} \tilde{\mathcal{N}} = T^*\mathcal{B} \arrow[dr] & = & G \overset{B}{\times} \mathfrak{u} \arrow[dl] \\ & \mathcal{B} & \end{tikzcd}$$

Example $G = SL_2$

$\mu : \tilde{\mathcal{N}} = T^*\mathbb{P}^1 \longrightarrow \mathcal{N}$

$\mu^{-1}(0) = \mathcal{B}_0 = \mathcal{B} = \mathbb{P}^1$

lemma The map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is very nice (proper and semismall).
($\tilde{\mathcal{N}}$ smooth)

For very nice maps, can show

$$\mu_*(\mathcal{L})[\dim \mathcal{N}] \in \mathrm{Perv}(\mathcal{N}, k)$$

$\mathcal{L}$ local system on $\tilde{\mathcal{N}}$

Corollary

$$\mu_*(\underline{k}_{\tilde{\mathcal{N}}})[\dim \mathcal{N}] \in \mathrm{Perv}(\mathcal{N}, k)$$

The Springer sheaf for G is

$$\mathrm{Spr} = \mathrm{Spr}_G := \mu_*(\underline{k}_{\tilde{\mathcal{N}}})[\dim \mathcal{N}]$$

in $\mathrm{Perv}_G(\mathcal{N}, k)$

goal: determine $\mathrm{End}(\mathrm{Spr})$

We will use the main diagram of Springer theory:

$$\begin{tikzcd} \tilde{\mathcal{G}}_{rs} \arrow[r, hook] \arrow[d, "\mu_{rs}"] \arrow[dr, phantom, "\lrcorner"] & \tilde{\mathcal{G}} \arrow[d, "\mu_{\mathcal{G}}"] & \tilde{\mathcal{N}} \arrow[l, hook'] \arrow[d, "\mu"] \\ \mathcal{G}_{rs} \arrow[r, hook] & \mathcal{G} & \mathcal{N} \arrow[l, hook'] \end{tikzcd}$$

where

$\mu : \tilde{\mathcal{N}} = G \times^B \mathfrak{u} \longrightarrow \mathcal{N} : \quad (g,x) \mapsto ad(g)\cdot x$

$\cap \qquad \cap$

$\mu_{\mathcal{G}} : \tilde{\mathcal{G}} := G \times^B \mathfrak{b} \longrightarrow \mathcal{G} : \quad (g,x) \mapsto ad(g)\, x$

$\cup \qquad \cup$

$\mu_{rs} : \tilde{\mathcal{G}}_{rs} = \mu_{rs}^{-1}(\mathcal{G}_{rs}) \rightarrow \mathcal{G}_{rs}$

$\| \qquad$ regular semisimple elements in $\mathcal{G}$

$G \times^B \mathfrak{b}_{rs}$

Example

$G = SL_n \quad \mathcal{G}_{rs}$: distinct eigenvalues

$\mu_{\mathcal{G}}$ is the Grothendieck-Springer simultaneous resolution. $\mu_{\mathcal{G}} : \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$

$$\begin{tikzcd}
\tilde{\mathfrak g}_{rs} \arrow[r, hook] \arrow[d, "\mu_{rs}"] & \tilde{\mathfrak g} \arrow[d, "\mu_{\mathfrak g}"] & \tilde{\mathcal N} \arrow[l, hook'] \arrow[d, "\mu"] \\
\mathfrak g_{rs} \arrow[r, hook] & \mathfrak g & \mathcal N \arrow[l, hook']
\end{tikzcd}$$

$\underbrace{\phantom{\mathfrak g_{rs}}}$ first study μ_{rs} $\underbrace{}$ then $\mu_{\mathfrak g}$ $\underbrace{}$ finally μ

lemma $W \circlearrowright$ $\mu_{rs} : \tilde{\mathfrak g}_{rs} \to \mathfrak g_{rs}$ is a normal covering map with Galois group W. $= \mathrm{Aut}_{\mathfrak g_{rs}}(\tilde{\mathfrak g}_{rs})$

Proof sketch Let's find the fibers:

$W \circlearrowright$

$$\tilde{\mathfrak g}_{rs} = G \times^B \mathfrak b_{rs} \simeq \left\{ (gB, y) \in \mathcal B \times \mathfrak g \;\middle|\; y \in \mathrm{ad}(g)(\mathfrak b) \right\}$$

$$\begin{tikzcd}
\tilde{\mathfrak g}_{rs} \arrow[dr, "\mu_{rs}"'] & & \{(gB,y)\} \arrow[dl, "\mathrm{pr}_2"] \\
& \mathfrak g &
\end{tikzcd}$$

$w \in N_G(T)$

$W = N_G(T)/T$

$W \circlearrowright$

$$\tilde{\mathfrak g}_{rs} = G \times^B \mathfrak b_{rs} \simeq G \times^T \mathfrak h_{rs}$$

$$w \cdot (g, \underset{\mathfrak h_{rs}}{\overset{}{x}}) \quad x \in \mathfrak h_{rs}$$

$$(g \cdot w^{-1}, \mathrm{ad}(w)x)$$

(Cartan): $\quad \mathfrak{h} := \overbrace{\mathfrak{g}_0^{y}}^{\text{centralizer}}$ is a Cartan subalgebra, and it is the only Cartan subalgebra containing $y \in \mathfrak{g}_{rs}$

$$\mu_{rs}^{-1}(y) = \{ gB \in \mathcal{B} \mid y \in \mathrm{ad}(g)(\mathfrak{b}) \}$$
$$= \{ \text{———} \mid u \in \text{unique Cartan sub of } \mathrm{ad}(g)(\mathfrak{b}) \}$$

hence: $\mu_{rs}^{-1}(y)$

$$= \{ \text{Borel subgroups } B^g \mid T \subset B^g \}$$

$$\begin{tikzcd} \{ \text{Borel subgroups } B^g \mid T \subset B^g \} & B^a \\ W = N_G(T)/T \arrow[u, "\simeq"] & a \arrow[u, mapsto] \end{tikzcd}$$

hence $\quad |\mu_{rs}^{-1}(y)| = |W|$

$\square$

$$\mu_{rs} : \tilde{G}_{rs} \xrightarrow{\text{normal covering}} G_{rs} \quad \text{with Galois group } W.$$

$\Rightarrow$ SES:

$$\pi_1(\tilde{G}_{rs}, x_0) \underset{\triangleleft}{\hookrightarrow} \pi_1(G_{rs}, x_0) \twoheadrightarrow W$$

$$\pi_1(G_{rs}, x_0) \circlearrowright M \qquad W \circlearrowright M$$

$\Rightarrow$ functor:

$$\begin{tikzcd}
L : k[W]\text{-mod}^{fg} \arrow[rr] \arrow[dr, hook, "\text{full \& faithful}"] & & \mathcal{L}oc^{lt}(G_{rs}, k) \\
& k[\pi_1(G_{rs})]\text{-mod}^{fg} \arrow[ur, "\simeq", "\text{monodromy}"'] &
\end{tikzcd}$$

$$M \mapsto L(M)$$

$$M \in k[\pi_1(G_{rs})]\text{-mod}^{fg} \qquad \mathcal{L} \leftrightarrow \mathcal{L}_x$$

Lemma $\quad \mu_{rs*}\,\underline{k}_{\tilde{G}_{rs}} = L(k[W])$

Corollary

$$\mathrm{End}(\mu_{rs*}\,\underline{k}_{\tilde{G}_{rs}}) = \mathrm{End}_{k[W]}(k[W]) = k[W]$$

Proof of lemma (sketch)

$$\begin{tikzcd}
\mathcal{L}oc(\tilde{\mathfrak{g}}_{rs}, k) \arrow[r, "\sim"] \arrow[d, "\mu_{rs *}"'] & k[\pi_1(\tilde{\mathfrak{g}}_{rs})]\text{-mod} \arrow[d, "\mathrm{Hom}_{k[\pi_1(\tilde{\mathfrak{g}}_{rs})]}(k[\pi_1(\mathfrak{g}_{rs})], -)"'] \\
\mathcal{L}oc(\mathfrak{g}_{rs}, k) \arrow[r, "\sim"] & k[\pi_1(\mathfrak{g}_{rs})]\text{-mod}
\end{tikzcd}$$

$$\mu_{rs *}(\underline{k}_{\tilde{\mathfrak{g}}_{rs}}) \xleftrightarrow{\text{monodromy}} \mathrm{Hom}_{k[\pi_1(\tilde{\mathfrak{g}})]}(k[\pi_1(\mathfrak{g})], k)$$

$$= k[W]$$

$$\begin{tikzcd}
\tilde{\mathcal{G}}_{rs} \arrow[r, hook] \arrow[d, "\mu_{rs}"] & \tilde{\mathcal{G}} \arrow[d, "\mu_{\mathcal{G}}"] & \tilde{\mathcal{N}} \arrow[l, hook'] \arrow[d, "\mu"] \\
\mathcal{G}_{rs} \arrow[r, hook] & \mathcal{G} & \mathcal{N} \arrow[l, hook']
\end{tikzcd}$$

$$\mu_{rs\,*}(\underline{k}_{\tilde{\mathcal{G}}_{rs}}) \simeq L(k[W]) \qquad \&$$

$$\mathrm{End}(\mu_{rs\,*}(\underline{k}_{\tilde{\mathcal{G}}_{rs}})) = k[W].$$

THEN

$$\mu_{\mathcal{G}*}(\underline{k}_{\tilde{\mathcal{G}}})[\dim \mathcal{G}] \simeq IC(\mathcal{G}_{rs}, L(k[W]))$$

$$\mathrm{End}(\mu_{\mathcal{G}*}(\underline{k}_{\tilde{\mathcal{G}}_{rs}})[\dim \mathcal{G}]) \underset{\downarrow}{=} k[W]$$

$IC(\mathcal{G}_{rs}, -)$ is fully faithful

THEN

$$\mathrm{Spr} = \mu_*(\underline{k}_{\tilde{\mathcal{N}}})[\dim \mathcal{N}] = IC(\mathcal{G}_{rs}, L(k[W]))\Big|_{\mathcal{N}}[\dots]$$

$$\mathrm{End}(\mathrm{Spr}) = k[W]$$

We get a functor

$$\mathrm{Hom}(\mathrm{Spr}, -) : \mathrm{Perv}_G(\mathcal{N}) \longrightarrow k[W]\text{-mod}$$

$\circlearrowleft k[W]$

$\Rightarrow$ study what this functor does to simple objects

[Juteau]

Theorem [Borho - MacPherson]

① If $IC(\mathcal{O}, \mathcal{L}) \in \mathrm{Perv}_G(\mathcal{N})$ is simple, then $\mathrm{Hom}(\mathrm{Spr}, IC(\mathcal{O}, \mathcal{L}))$ is 0 or an irreducible $k[W]$-module.

② Every irreducible $k[W]$-module V arises as $\mathrm{Hom}(\mathrm{Spr}, IC(\mathcal{O}, \mathcal{L}))$ for a unique simple $IC(\mathcal{O}, \mathcal{L})$.

We hence get an injective map

$$\nu : \operatorname{Irr}(k[W]) \longrightarrow \operatorname{Irr}(\operatorname{Perv}_G(\mathcal{N}, k))$$
$$V \longmapsto IC(\mathcal{O}, \mathcal{L})$$

$\exists! (\mathcal{O}, \mathcal{L}) : \operatorname{Hom}(Spr, IC(\mathcal{O}, \mathcal{L})) = V$

This is the springer correspondence.

A perverse sheaf in the image of ν

is said to occur in the Springer correspondence

Proof sketch ①

Let $h : \mathcal{N} \hookrightarrow G$. Let $\mathcal{F} = IC(\mathcal{O}, \mathcal{L})$ simple.

Suppose $\operatorname{Hom}(Spr, \mathcal{F}) \neq 0$.

$$Spr[\ldots] \simeq h^* IC(G_{rs}, L(k[W]))$$

so get a nonzero map $IC(G_{rs}, L(k[W])) \twoheadrightarrow \underbrace{h_* \mathcal{F}}_{\text{simple}}$

restrict to G_{rs}:

$$L(k[W])[\ldots] \twoheadrightarrow h_* \mathcal{F}|_{G_{rs}}$$

local systems are closed under quotients

$\Rightarrow h_* \mathcal{F}|_{G_{rs}}$ is a shifted local system

of the form $L(V)[\ldots]$ of some

simple quotient V of $k[W]$.

$\Rightarrow h_* \mathcal{F} = IC(G_{rs}, L(V))$

and $\mathrm{Hom}(Spr, \mathcal{F}) = \mathrm{Hom}(h^* IC(G_{rs}, L(k[W]), \mathcal{F})$

$= \mathrm{Hom}(IC(G_{rs}, L(k[W]), IC(G_{rs}, L(V)))$

$= \mathrm{Hom}(L(k[W], L(V))= \mathrm{Hom}_{k[W]}(k[W], V) = V.$

② For $V \in \mathrm{Irr}(k[W])$, take $\mathcal{F}$ with $h_* \mathcal{F} = IC(G_{rs}, L(V))$.

Then $v(\mathcal{F}) = V$ as above.

Corollary

$$\mathrm{Spr} \cong \bigoplus_{\substack{\mathcal{O} \subset \mathcal{N} \\ \mathcal{L} \text{ irreducible} \\ \text{local system on } \mathcal{O}}} IC(\mathcal{O}, \mathcal{L})^{\oplus m_{\mathcal{O},\mathcal{L}}}$$

$\dim V_{\mathcal{O},\mathcal{L}} = m_{\mathcal{O},\mathcal{L}}$

$$\cong \bigoplus_{\mathcal{O} \subset \mathcal{N}} IC(\mathcal{O}, \mathcal{L}) \otimes V_{\mathcal{O},\mathcal{L}}$$

How does this relate to the original theorem by Springer?

Recall $\mathcal{B}_x \subset \mathcal{B}$ closed subvariety for every $x \in \mathcal{N}$

For $x \in \mathcal{N}$, let

$$A_G(x) := G^x / (G^x)^\circ$$ (G^x: stabilizer)

Let $\mathcal{O}$ be the orbit in $\mathcal{N}$ of x. Then

$$\mathrm{Loc}_G^{ft}(\mathcal{O}) \simeq A_G(x)\text{-mod}$$
$$\mathcal{L} \longmapsto \mathcal{L}_x$$

(equivariant version of the monodromy equivalence)

Theorem [Springer] Let $x \in \mathcal{N}$.

$H^i(\mathcal{B}_x, \mathbb{Q})$ is equipped with a natural action of $A_G(x) \times W$, so that

$$H^i(\mathcal{B}_x, \mathbb{Q}) \simeq \bigoplus_{L \in \mathrm{Irr}(A_G(x))} L \otimes V^i_{x,L}$$

where each $V^i_{x,L}$ is a $\mathbb{Q}[W]$-module.

Moreover: ① $V^{2\dim \mathcal{B}_x}_{x,L}$ is irr. or 0

② each $V \in \mathrm{Irr}(\mathbb{Q}[W])$ occurs as $V^{2\dim \mathcal{B}_x}_{x,L}$ for some x and L.

Proof sketch

Use $\mathrm{Spr}_x \simeq R\Gamma(\underline{\mathbb{Q}}_{\mathcal{B}_x})[\dim \mathcal{N}]$

$$\Rightarrow \quad \mathrm{IH}^i(\mathcal{B}_x, \mathbb{Q}) = H^{i-\dim\mathcal{N}}(\mathrm{Spr}_x)$$

$$= \bigoplus_{\mathcal{O},\mathcal{L}} \underbrace{H^{i-\dim\mathcal{N}}(IC(\mathcal{O},\mathcal{L}))_x}\otimes V_{\mathcal{O},L} \circlearrowleft W$$

G-equivariant local system on an orbit of x

$\Rightarrow$ its stalk is a $A_G(x)$-module

$$= \bigoplus_{L\in \mathrm{Irr}(A_G(x))} L \otimes V^i_{x,L}$$

The Springer correspondence

$\nu : \mathrm{Irr}(k[W]) \longrightarrow \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}, k))$

is not surjective. How can we reach all the simples in the codomain?

[Lusztig] [Achar, Henderson, Juteau, Riche]

Theorem There is a bijection

$$\nu : \bigsqcup_{[L, \mathcal{O}, \mathcal{L}]} \mathrm{Irr}[k[N_G(L)/L]] \xrightarrow{\sim} \mathrm{Irr}\, \mathrm{Perv}_G(\mathcal{N}, k)$$

$\begin{cases} L \subset G & \text{Levi subgroup} \\ \mathcal{O} \subset \mathcal{N}_L & \text{orbit} \\ \mathcal{L} \in \mathrm{Irr}(\mathrm{Loc}^{ft}_L(\mathcal{O}, k)) & \end{cases}$

$\mathrm{Irr}(k[W])$

$[L, \mathcal{O}, \mathcal{L}] = [T, \{0\}, k]$

This is the generalized Springer correspondence