Example

G finite group, k field.

Suppose $\operatorname{char} k = p$ and $p \mid |G|$.

Often, G has wild representation type!

$$\underset{kG\text{-mod}}{\overset{\shortparallel}{\operatorname{Rep} G}} \text{ too hard to understand...}$$

We already understand the projective kG-modules, so let's quotient them out!

$$\operatorname{Stab} G = kG\text{-mod} \big/ kG\text{-proj}$$

→ this is the "right place" to think about group cohomology...

Stab G is no longer abelian ...

but the exact sequences in Rep G

still play an important role in Stab G.

$$\mathrm{Rep}\,G \longrightarrow \mathrm{Stab}\,G \xrightarrow{H^i} k\text{-Vec} \qquad i \geq 1$$

SES

$$X \hookrightarrow Y \twoheadrightarrow Z \longmapsto \; ? \; \longmapsto \text{long exact sequence}$$

$$\cdots \to H^i(X) \to H^i(Y) \to H^i(Z) \to H^{i+1}(X) \to \cdots$$

Example: Homotopy category

Let $\mathcal{A}$ be an abelian category.

for example, $\mathcal{A} = k\text{-vec}$ (k field)

$\mathcal{A} = R\text{-mod}$ (R ring)

$\mathcal{A} = Ab$

$\mathcal{A} = G\text{-Rep}$ (G group)

$\mathcal{A} = QC(X)$ (X scheme)

$C(\mathcal{A})$ is the category of complexes over $\mathcal{A}$

$$\cdots \longrightarrow X^{-1} \xrightarrow{d} X^{0} \xrightarrow{d} X^{1} \xrightarrow{d} X^{2} \xrightarrow{d} \cdots \qquad d^2 = 0$$

Two morphisms $f, g : X^\bullet \longrightarrow Y^\bullet$ in $C(\mathcal{A})$ are homotopic if there exist morphisms $X^i \xrightarrow{h} Y^{i-1}$ such that $f - g = dh + hd$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-1} \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, dashed] & X^{0} \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, dashed, "h"'] & X^{1} \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, dashed, "h"'] & \cdots \arrow[dl, dashed] \\
\cdots \arrow[r] & Y^{-1} \arrow[r, "d"'] & Y^{0} \arrow[r, "d"'] & Y^{1} \arrow[r] & \cdots
\end{tikzcd}$$

The <u>homotopy category</u> $K(A)$ is the category whose objects are complexes over A and whose morphisms are

$$\operatorname{Hom}_{K(A)}(X^\bullet, Y^\bullet) = \operatorname{Hom}_{C(A)}(X^\bullet, Y^\bullet) \Big/ \underset{\text{homotopy}}{\sim}$$

The <u>translation functor</u> Σ is the functor "give a push to the left".

specifically:

$$\Sigma\Big(\cdots X^{-1} \xrightarrow{d} \overbrace{X^0}^{\text{degree }0} \xrightarrow{d} X^1 \xrightarrow{d} X^2 \longrightarrow \cdots\Big)$$

$$= \Big(\cdots X^{-1} \xrightarrow{-d} X^0 \xrightarrow{-d} \underbrace{X^1}_{\text{degree }0} \xrightarrow{-d} X^2 \longrightarrow \cdots\Big)$$

Again, $K(\mathcal{A})$ is no longer abelian...

but (split) short exact sequences in $C(\mathcal{A})$ are still important!

$$\begin{tikzcd} C(\mathcal{A}) \arrow[r] & K(\mathcal{A}) \arrow[r, "H^i"] & \mathcal{A} \\ \text{SES} \arrow[r, mapsto] \arrow[rr, bend right, mapsto] & ? \arrow[r, mapsto] & \text{long exact sequence} \end{tikzcd}$$

Every morphism $X^\bullet \xrightarrow{f^\bullet} Y^\bullet$ in $C(\mathcal{A})$ has a <u>cone</u> $C_f^\bullet$ in $C(\mathcal{A})$ with

$\boxed{C_f^i = Y^i \oplus X^{i+1}}$ and differential

$$d_{C_f} : Y^i \oplus X^{i+1} \xrightarrow{\begin{bmatrix} d & f^{i+1} \\ 0 & \pm d \end{bmatrix}} Y^{i+1} \oplus X^{i+2}.$$

<u>exercise</u> $C_f^\bullet$ is a complex

We find morphisms in $C(\mathcal{A})$

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \longrightarrow C_f^\bullet \longrightarrow \Sigma X^\bullet \quad (*)$$

given degreewise by

$$X^i \xrightarrow{f^i} Y^i \hookrightarrow Y^i \oplus X^{i+1} \twoheadrightarrow \underset{(\Sigma X)^i}{X^{i+1}}$$

<u>exercise</u> morphisms of complexes

a diagram of the form

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow \Sigma A^\bullet$$

in $K(\mathcal{A})$ is called an <u>exact triangle</u> if it is isomorphic to (*) for some $X^\bullet \xrightarrow{f} Y^\bullet$.

Specifically:

- it is exact if we can find isomorphisms in $K(\mathcal{A})$ such that:

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, "\sim"] & B^\bullet \arrow[r] \arrow[d, "\sim"] & C^\bullet \arrow[r] \arrow[d, "\sim"] & \Sigma A^\bullet \arrow[d, "\sim"] \\
X^\bullet \arrow[r, "f"'] & Y^\bullet \arrow[r] & C_f^\bullet \arrow[r] & \Sigma X^\bullet
\end{tikzcd}$$

(*)

Remark

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, "\sim \text{ in } K(A)"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "id"'] & X \arrow[r] & C_{id} \arrow[r] & \Sigma X
\end{tikzcd}$$

is an exact Δ (*)

<u>Proof</u> need to show $C^{\bullet}_{id} \simeq 0$ in $K(A)$

so need to find

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \oplus X^{i} \arrow[r, "\begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}"] \arrow[d, "id"] & X^{i} \oplus X^{i+1} \arrow[r, "\begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}"] \arrow[d, "id"] \arrow[dl, dashed, "h"'] & X^{i+1} \oplus X^{i+2} \arrow[r] \arrow[d, "id"] \arrow[dl, dashed, "h"'] & \cdots \\
\cdots \arrow[r] & X^{i-1} \oplus X^{i} \arrow[r, "\begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}"'] & X^{i} \oplus X^{i+1} \arrow[r, "\begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}"'] & X^{i+1} \oplus X^{i+2} \arrow[r] & \cdots
\end{tikzcd}$$

Take $h = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} X^{i} \oplus X^{i+1} \longrightarrow X^{i-1} \oplus X^{i}$.

Theorem

An exact triangle in $K(\mathcal{A})$

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} \Sigma X^\bullet$$

induces a long exact sequence in $\mathcal{A}$:

$$\cdots \to H^n(X) \xrightarrow{H^n(u)} H^n(Y) \xrightarrow{H^n(v)} H^n(Z) \xrightarrow{H^n(w)} H^{n+1}(X) \xrightarrow{H^{n+1}(u)} H^{n+1}(Y) \xrightarrow{H^{n+1}(v)} H^{n+1}(Z) \to \cdots$$

$\boxed{\text{⚠ main reason we care about exact triangles}}$

$Y \to Z \to \Sigma X \xmapsto{H^\bullet}$ same long exact sequence

$Z \to \Sigma X \to \Sigma Y \xmapsto{H^\bullet}$ same long exact sequence

Remark

Consider

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & K(\mathcal{A}) & \longrightarrow & \mathcal{D}(\mathcal{A}) \\ \text{SES} & \longmapsto & & \longmapsto & \text{exact}\,\Delta \\ X & \longmapsto & (\cdots \to 0 \to X \to 0 \to \cdots) & & \end{array}$$

This functor sends <u>split</u> short exact sequences in $\mathcal{A}$ to exact triangles in $K(\mathcal{A})$! (exercise)

What about non-split SES ?

It doesn't ...

$$\boxed{\text{This is why people often work in the derived category } \mathcal{D}(\mathcal{A}).}$$

$X \overset{f}{\hookrightarrow} Y \twoheadrightarrow Z$ SES in $\mathcal{A}$

gives

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] \arrow[d] & Y^\bullet \arrow[r] \arrow[d] & C_f \arrow[r] \arrow[d, dashed, "\varphi"] & {} & \text{exact } \Delta \\
X^\bullet \arrow[r, "f"'] & Y^\bullet \arrow[r] & Z \arrow[r] & {} & \text{not necessarily exact in } K(\mathcal{A})
\end{tikzcd}$$

φ is not necessarily an iso in $K(\mathcal{A})$,

but it is **always** a quasi-isomorphism.

Triangulated categories

A triangulated category $\mathcal{T}$ is an additive category with

- a translation functor ("suspension")

$$\Sigma : \mathcal{T} \longrightarrow \mathcal{T} \quad \text{auto-equivalence}$$
$$X \longmapsto \Sigma X = X[1]$$

- a class of triangles (called exact)

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow \Sigma X^\bullet$$

satisfying axioms TR1 - TR4

$\boxed{\text{TR 1}}$ BOOKKEEPING AXIOM

- $\forall X \in \mathcal{T}$: $X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow \Sigma X$ is exact

- $\forall X \xrightarrow{u} Y$ fits into an exact triangle

$$X \xrightarrow{u} Y \xrightarrow{v} \underbrace{Z}_{C_u \text{ cone of } u} \xrightarrow{w} \Sigma X$$

- every triangle isomorphic to an exact triangle is exact.

$\boxed{\text{TR 2}}$ ROTATION AXIOM

If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is exact,

then so are $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$

and $\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} w} X \xrightarrow{u} Y \xrightarrow{v} Z$

$\boxed{\text{TR3}}$ COMPLETION AXIOM

given

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] & \Sigma X & \text{exact} \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma Z' & \text{exact}
\end{tikzcd}$$

there exists h making everything commute:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dashed, "h"] & \Sigma X \arrow[d, dashed, "\Sigma f"] & \text{exact} \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma Z' & \text{exact}
\end{tikzcd}$$

$\boxed{\text{TR 4}}$ OCTAHEDRAL AXIOM

given
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "vu"'] & Y \arrow[d, "v"] \\ & Z \end{tikzcd}$$

how do the cones of u, v, vu relate?

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "vu"'] & Y \arrow[r] \arrow[d, "v"] & C_u \arrow[r] \arrow[d, dashed] & {} \\ & Z \arrow[r] \arrow[d] & C_{vu} \arrow[r] \arrow[dl, dashed] & {} \\ & C_v \arrow[d] & & \\ & {} & & \end{tikzcd}$$

$$C_u \longrightarrow C_{vu} \longrightarrow C_v \longrightarrow$$

is an exact triangle

$$
\begin{tikzcd}
U \arrow[r, hook] \arrow[dr] & W \arrow[r, two heads] \arrow[d] & W/U \arrow[dd] \\
 & X \arrow[ddr, phantom] \arrow[dr, two heads] \arrow[dd, two heads] & \\
 & & X/U \arrow[dl, two heads] \\
 & X/W &
\end{tikzcd}
$$

third iso theorem:

$$\frac{X/U}{W/U} = X/W$$

Facts

- TR 3 follows from TR1, TR2, TR4
- If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ exact, then $vu=0$ $wv=0$ $(\Sigma u)w=0$
- "monomorphism" & "epimorphism" are not good notions in $\mathcal{T}$:

  every $X \xhookrightarrow{u} Y$ splits: $Y \simeq X \oplus C_u$

  every $Y \xrightarrow{u} Z$ (epi) splits: $Y \simeq Z \oplus \Sigma^{-1} C_u$
- every morphism $X \xrightarrow{u} Y$ has a cone, unique up to (non unique ⚠) isomorphism.

Theorem

$K(\mathcal{A})$ is a triangulated category.

Theorem

Stab G is triangulated, with

$$\Sigma : \mathrm{Stab}\, G \longrightarrow \mathrm{Stab}\, G$$
$$X \longmapsto \Sigma X$$

$\Sigma^{-1} = \Omega$

$\in kG\text{-mod}$ $X \hookrightarrow I \twoheadrightarrow \Sigma X$ cokernel

injective hull

and a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$

in Stab G is exact if and only if

$\exists$ short exact sequences in Rep G

$$0 \hookrightarrow X \xrightarrow{u'} Y \oplus \mathrm{proj} \xrightarrow{v'} Z \twoheadrightarrow 0 \quad \&$$

$$0 \hookrightarrow Y \xrightarrow{v''} Z \oplus \mathrm{proj} \xrightarrow{w''} \Sigma X \twoheadrightarrow 0$$

with

$$\mathrm{Rep}\, G \longrightarrow \mathrm{Stab}\, G.$$
$$u', v', v'', w'' \longmapsto u, v, v, w$$