

Example G finite group, k field.

Suppose $\text{char } k = p$ and $p \mid |G|$.

Often, G has wild representation type!

$\text{Rep } G$ too hard to understand...
 $\overset{\text{"}}{kG\text{-mod}}$

We already understand the
injective kG -modules, so let's
quotient them out!

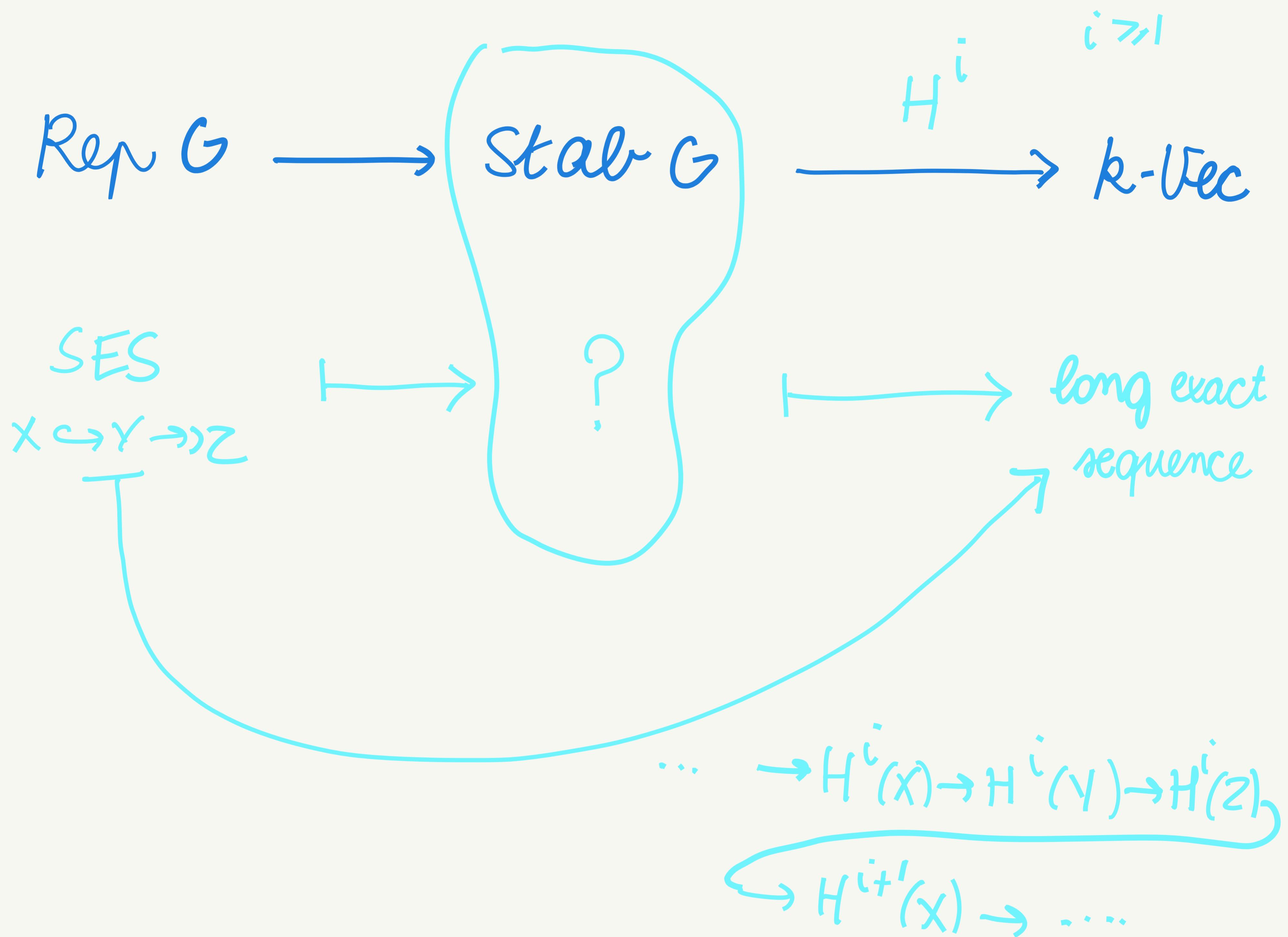
$$\text{Stab } G = \frac{kG\text{-Mod}}{kG\text{-proj}}$$

→ this is the "right place" to think
about group cohomology ...

$\text{Stab } G$ is no longer abelian ...

but the exact sequences in $\text{Rep } G$

still play an important role in $\text{Stab } G$.



Example: Homotopy category

Let A be an abelian category.

for example, $A = k\text{-vec}$ (k field)

$A = R\text{-mod}$ (R ring)

$A = \text{Ab}$

$A = G\text{-Rep}$ (G group)

$A = \text{QC}(X)$ (X scheme)

$C(A)$ is the category of complexes over A

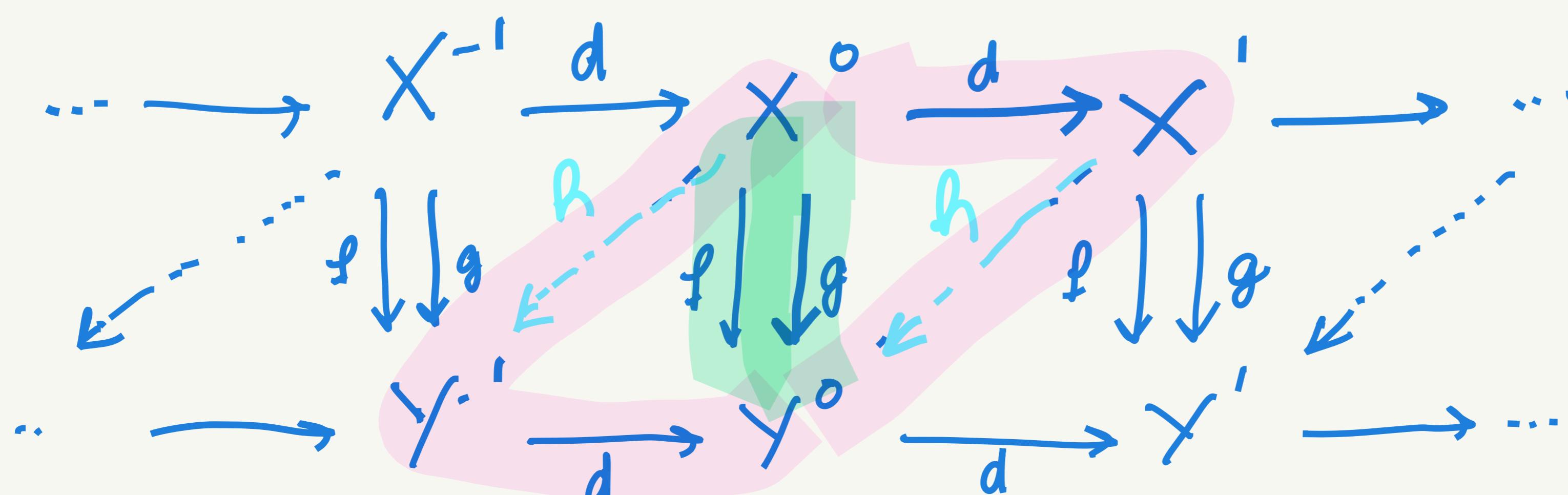
$$\dots \rightarrow X^{-1} \xrightarrow{d} X^0 \xrightarrow{d} X^1 \xrightarrow{d} X^2 \xrightarrow{d} \dots$$

$d^2 = 0$

Two morphisms $f, g: X^\bullet \rightarrow Y^\bullet$ in $C(A)$

are homotopic if there exist morphisms

$$X^i \xrightarrow{h} Y^{i-1} \quad \text{such that} \quad f - g = dh + hd$$



The homotopy category $K(A)$ is the category whose objects are complexes over A and whose morphisms are

$$\text{Hom}_{K(A)}(X^\bullet, Y^\bullet) = \frac{\text{Hom}(X^\bullet, Y^\bullet)}{\sim} \quad \text{Homotopy}$$

The translation functor Σ is the functor "give a push to the left".

specifically:

$$\Sigma \left(\dots X^{-1} \xrightarrow{d} X^0 \xrightarrow{d} X^1 \xrightarrow{d} X^2 \xrightarrow{\dots} \right)$$

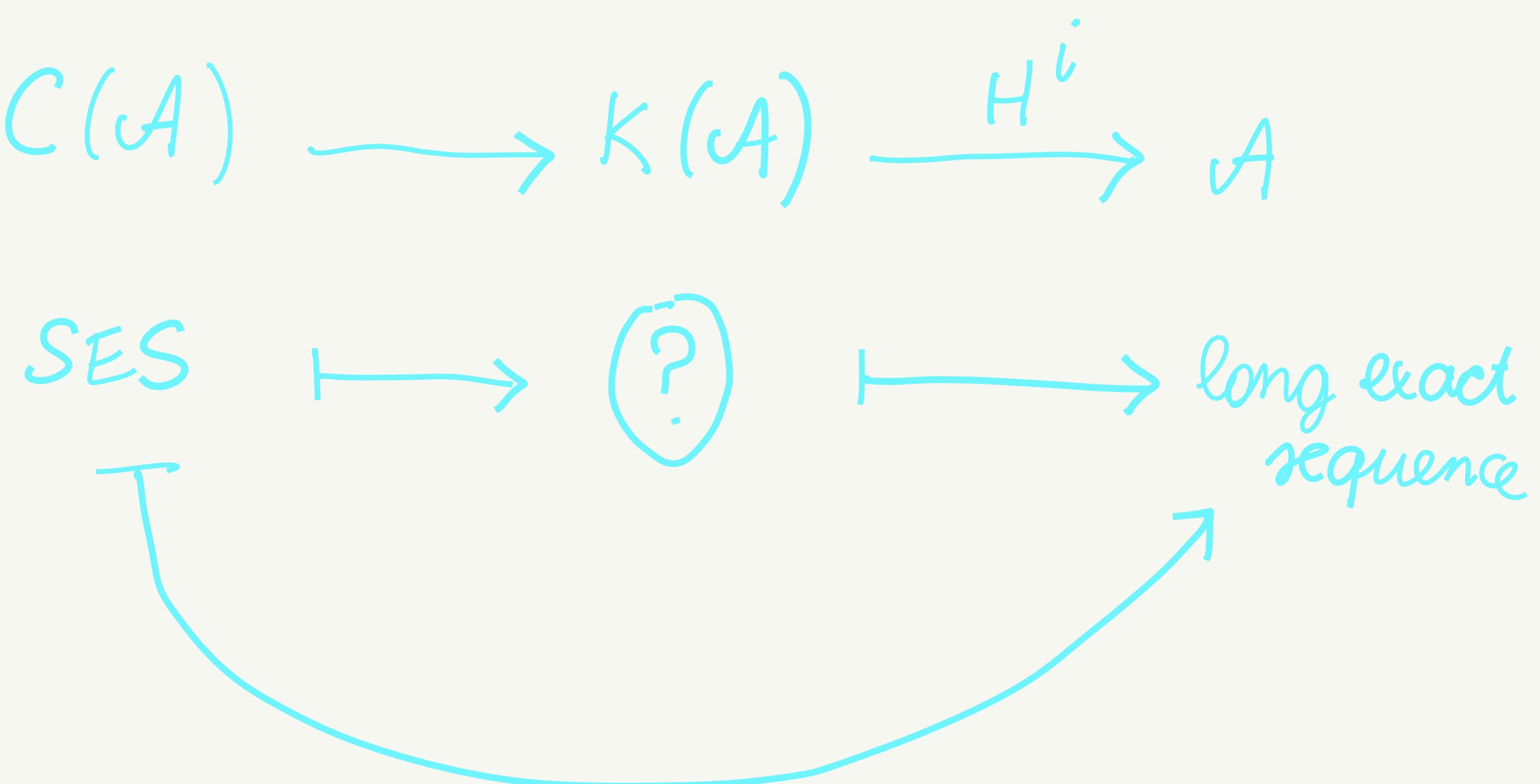
degree 0

$$= \left(\dots X^{-1} \xrightarrow{-d} X^0 \xrightarrow{-d} X^1 \xrightarrow{-d} X^2 \xrightarrow{\dots} \right)$$

degree 0

Again, $K(A)$ is no longer abelian ...

but (split) short exact sequences in
 $C(A)$ are still important!



Every morphism $X^\bullet \xrightarrow{f^\bullet} Y^\bullet$ in $C(A)$

has a cone C_f^\bullet in $C(A)$ with

$$C_f^i = Y^i \oplus X^{i+1}$$

and differential

$$d: Y^i \oplus X^{i+1} \xrightarrow{\begin{bmatrix} d & f^{i+1} \\ 0 & \pm d \end{bmatrix}} Y^{i+1} \oplus X^{i+2}.$$

exercise C_f^\bullet is a complex

We find morphisms in $C(A)$

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \longrightarrow C_f^\bullet \longrightarrow \Sigma X^\bullet \quad (*)$$

given degreewise by

$$x^i \xrightarrow{f^i} y^i \hookrightarrow Y^i \oplus X^{i+1} \twoheadrightarrow X^{i+1} \quad (\Sigma X)^i$$

exercise morphisms of complexes

a diagram of the form

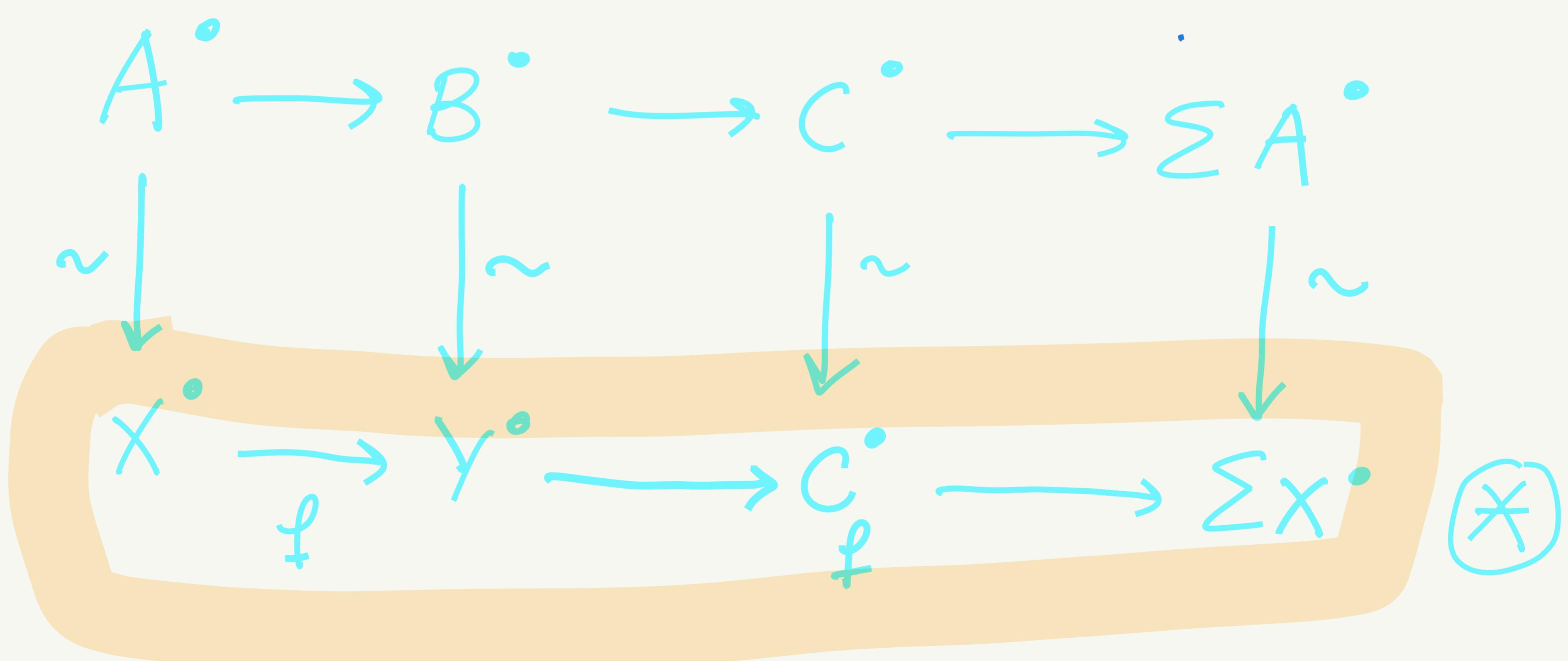
$$A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow \Sigma A^\bullet$$

in $K(A)$ is called **an exact triangle**

if it is isomorphic to \circledast for some $X^\bullet \xrightarrow{f} Y^\bullet$.

Specifically:

- it is exact if we can find isomorphisms in $K(A)$ such that:



Remark

$$\begin{array}{ccccccc}
 X & \xrightarrow{\text{id}} & X & \longrightarrow & O & \longrightarrow & \Sigma X \text{ is an} \\
 // & & // & & \downarrow \sim \text{in } K(A) & // & \text{exact } \Delta \\
 X & \xrightarrow{\text{id}} & X & \longrightarrow & C_{\text{id}} & \longrightarrow & \Sigma X \quad (*) \\
 & & & & & &
 \end{array}$$

Proof need to show $C_{\text{id}}^* \approx 0$ in $K(A)$

so need to find

$$\begin{array}{ccccc}
 \dots & \rightarrow & X^{i-1} \oplus X^i & \xrightarrow{[d \quad d]} & X^i \oplus X^{i+1} \\
 & & \downarrow \text{id} & & \downarrow \text{id} \\
 \dots & \rightarrow & X^{i-1} \oplus X^i & \xrightarrow{[d \quad d]} & X^i \oplus X^{i+1} \\
 & & & & \xrightarrow{[d \quad d]} X^{i+1} \oplus X^{i+2} \rightarrow \dots
 \end{array}$$

Take $h = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $X^i \oplus X^{i+1} \xrightarrow{[d \quad d]} X^{i-1} \oplus X^i$.

Theorem

an exact triangle in $K(A)$

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} \Sigma X^\bullet$$

induces a long exact sequence in A :

$$\dots \rightarrow H^n(X) \xrightarrow{H^n(u)} H^n(Y) \xrightarrow{H^n(v)} H^n(Z) \xrightarrow{H^n(w)} H^{n+1}(X)$$

\curvearrowright

$$H^{n+1}(u) : H^{n+1}(Y) \rightarrow H^{n+1}(Z) \rightarrow \dots$$

↙ main reason we care
about exact triangles

$$Y \rightarrow Z \rightarrow \Sigma X \xrightarrow{H^\bullet} \text{same long exact sequence}$$
$$Z \rightarrow \Sigma X \rightarrow \Sigma Y \xrightarrow{H^\bullet} \text{same long exact sequence}$$

Remark

Consider

$$\begin{array}{ccc} A & \xrightarrow{\quad} & K(A) \\ SES & \xrightarrow{\quad} & \xrightarrow{\quad} \mathcal{D}(A) \\ X & \xrightarrow{\quad} & \xrightarrow{\quad} \text{exact} \Delta \\ & & (\cdots \rightarrow 0 \rightarrow X \rightarrow 0 \rightarrow \cdots) \end{array}$$

This functor sends split short exact sequences in A to exact triangles in $K(A)$! (exercise)

What about non-split SES ?

It doesn't ...

This is why people often work
in the derived category $\mathcal{D}(A)$.

$X \xrightarrow{f} Y \rightarrow Z$ SES in \mathcal{A}

gives

$$\begin{array}{ccccccc} X^{\circ} & \xrightarrow{f^{\circ}} & Y^{\circ} & \rightarrow & C_f & \rightarrow & \text{exact } \Delta \\ \downarrow & & \downarrow & & \downarrow \varphi & & \\ X^{\circ} & \xrightarrow{f^{\circ}} & Y^{\circ} & \rightarrow & Z & \rightarrow & \text{not necessarily exact in } K(\mathcal{A}) \end{array}$$

φ is not necessarily an iso in $K(\mathcal{A})$,

but it is always a quasi-isomorphism.

Triangulated categories

A triangulated category \mathcal{T} is an additive category with

- a translation functor ("suspension")

$$\Sigma : \mathcal{T} \longrightarrow \mathcal{T} \quad \text{auto-equivalence}$$

$$X \longmapsto \Sigma X = X[1]$$

- a class of triangles (called exact)

$$X^\circ \longrightarrow Y^\circ \longrightarrow Z^\circ \longrightarrow \Sigma X^\circ$$

satisfying axioms TR1 - TR4

TR 1 BOOKKEEPING AXIOM

- $\forall X \in \mathcal{T}: X \xrightarrow{\text{id}_X} X \longrightarrow 0 \longrightarrow \Sigma X$ is exact

- $\forall X \xrightarrow{u} Y$ fits into an exact triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

C_u cone of u

- every triangle isomorphic to an exact triangle is exact.

TR 2

ROTATION Axiom

If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is exact,

then so are $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$

and $\Sigma Z \xrightarrow{-\Sigma w} X \xrightarrow{u} Y \xrightarrow{v} Z$

TR 3

COMPLETION Axiom

given

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ f \downarrow & \circ & \downarrow g & & & & \text{exact} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma Z' \end{array}$$

exact

there exists h making everything commute:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ f \downarrow & & \downarrow g & & \text{---} & & \text{---} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma Z' \end{array}$$

exact

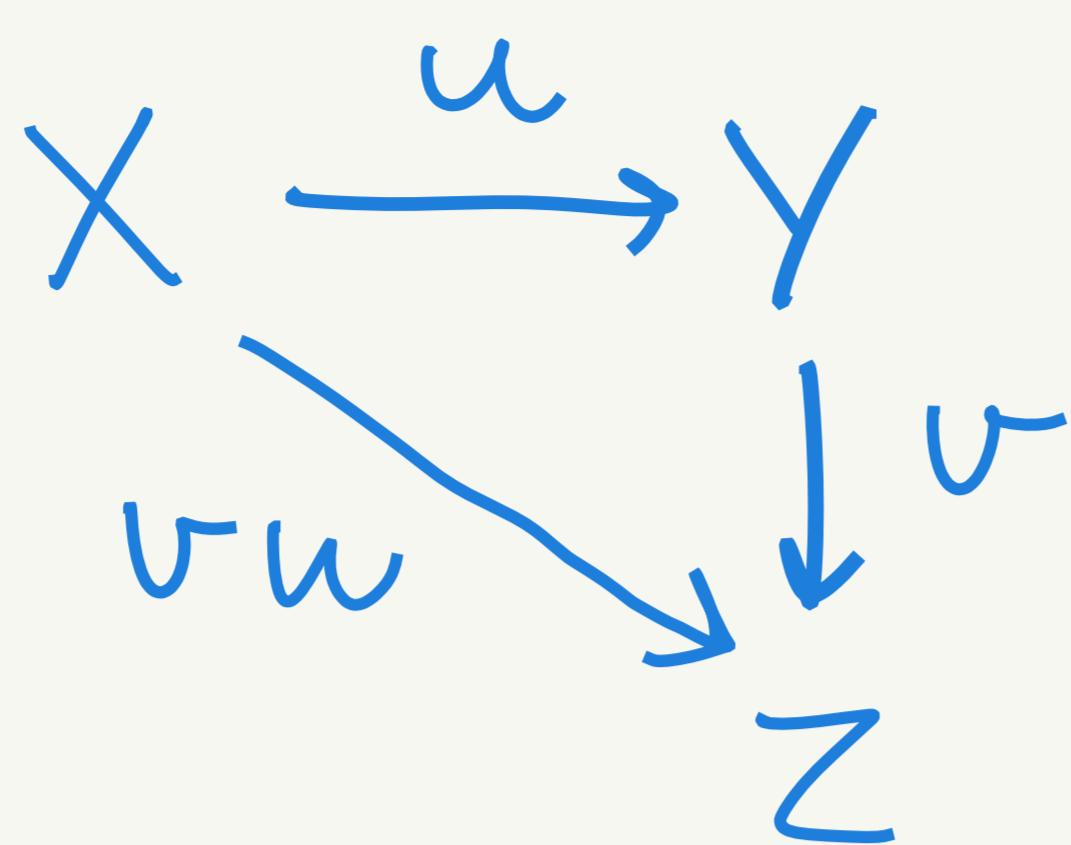
h

Σf

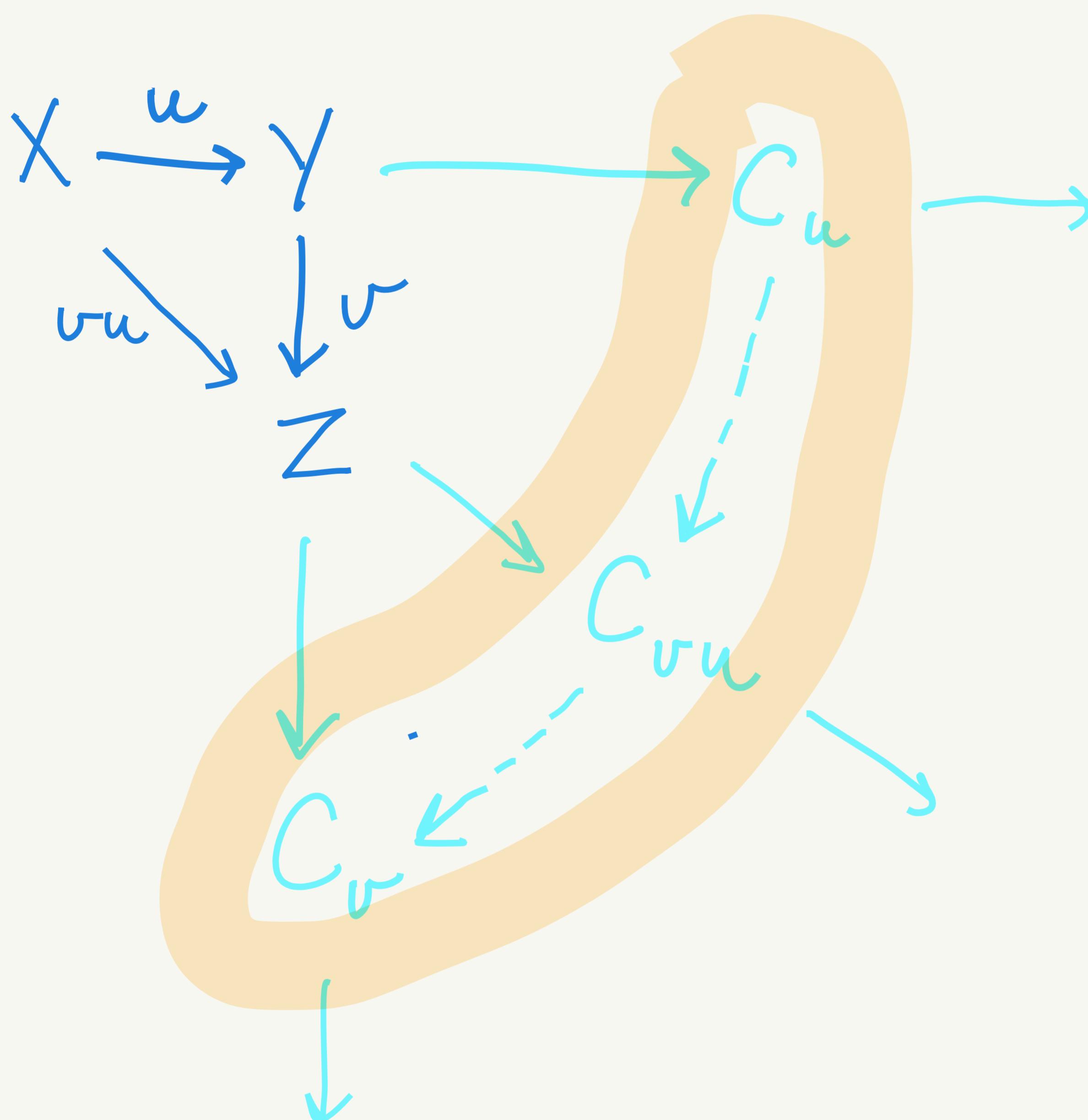
TR 4

OCTAHEDRAL AxiOM

given

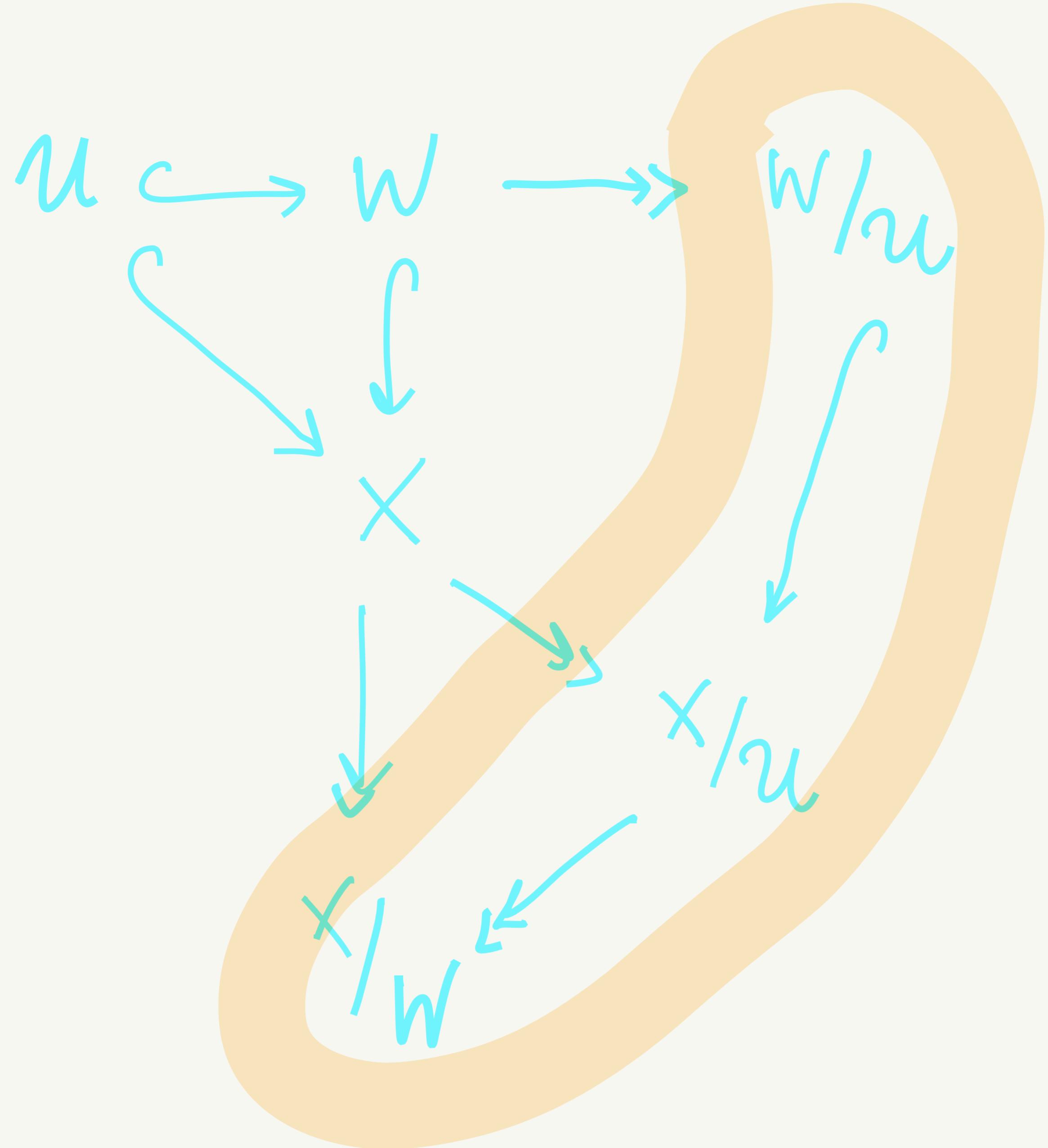


how do the cones of u, v, vu relate?



$$C_u \rightarrow C_{vu} \rightarrow C_v \rightarrow$$

is an exact triangle



third iso theorem :

$$\frac{X/u}{w/u} = \frac{X}{w}$$

Facts

- TR 3 follows from TR1, TR2, TR4
- If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \sum X$ exact,
then $vu = 0$ $wv = 0$ $(\sum u) w = 0$
- "monomorphism" & "epimorphism"
are not good notions in \mathcal{T} :
every $X \hookrightarrow Y$ splits: $Y \simeq X \oplus C_u$
every $Y \xrightarrow{u} Z$ splits : $Y \simeq Z \oplus \sum \tilde{C}_u$
- every morphism $X \xrightarrow{u} Y$ has a cone,
unique up to (non unique Δ)
isomorphism.

Theorem

$K(A)$ is a triangulated category.

~~to do~~.

Theorem

$\text{Stab } G$ is triangulated, with

$$\Sigma : \text{Stab } G \rightarrow \text{Stab } G$$

$$\begin{array}{ccc} \Sigma^{-1} = \Omega & X & \xrightarrow{\quad} \Sigma X \\ & \xleftarrow{\text{in } kG\text{-mod}} X \hookrightarrow I \xrightarrow{\text{cokernel}} \Sigma X \\ & & \text{injective hull} \end{array}$$

and a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$.

in $\text{Stab } G$ is exact if and only if

\exists short exact sequences in $\text{Rep } G$

$$0 \hookrightarrow X \xrightarrow{u'} Y \oplus \text{proj} \xrightarrow{v'} Z \twoheadrightarrow 0 \quad \&$$

$$0 \hookrightarrow Y \xrightarrow{v''} Z \oplus \text{proj} \xrightarrow{w''} \Sigma X \twoheadrightarrow 0$$

with

$$\text{Rep } G \longrightarrow \text{Stab } G.$$

$$u', v', v'', w'' \longmapsto u, \sigma, v, w$$