- Truncations of derived categories
- t- Structures and hearts
- t- Exact Functors
- Some Comments

· Truncations

Let $\mathcal{A}$ be an abelian category. Let $A^\cdot$ be a cochain complex.

There are four ways to truncate the complex $A^\cdot$:

① The stupid truncation $\sigma_{\geq n} A^\cdot$ defined by

$$(\sigma_{\geq n} A^\cdot)^i = \begin{cases} 0 & \text{if } i < n \\ A^i & \text{if } i \geq n \end{cases}$$

$$\begin{tikzcd} \sigma_{\geq n} A^\cdot = \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \cdots \\ A^\cdot = & \cdots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

② The stupid truncation $\sigma_{\leq n} A^\cdot$ defined by

$$(\sigma_{\leq n} A^\cdot)^i = \begin{cases} 0 & \text{if } i > n \\ A^i & \text{if } i \leq n \end{cases}$$

$$\begin{tikzcd} A^\cdot = \arrow[d] & \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \cdots \\ \sigma_{\leq n} A^\cdot = & \cdots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

③ The canonical truncation $\tau_{\leq n} A^\cdot$ defined by

$$(\tau_{\leq n} A^\cdot)^i = \begin{cases} A^i & i < n \\ \ker(d^n) & i = n \\ 0 & i > n \end{cases}$$

$$\begin{tikzcd} \tau_{\leq n} A^\cdot = \arrow[d, "i"'] & \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & \ker d^n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ A^\cdot = & \cdots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

④ The canonical truncation $\tau_{\geq n} A^\bullet$ defined by

$$(\tau_{\geq n} A^\bullet)^i = \begin{cases} A^n & i>n \\ \mathrm{coker}(d^{n-1}) & i=n \\ 0 & i<n \end{cases}$$

$$\begin{tikzcd}
A^\bullet = \arrow[d, "q"'] & \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \cdots \\
\tau_{\geq n} A^\bullet = & \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d^{n-1}) \arrow[r] & A^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Easy fact: $\sigma_{\geq n}$, $\sigma_{\leq n}$, $\tau_{\leq n}$, $\tau_{\geq n}$ induce functors: $C^\bullet(\mathcal{A}) \to C^\bullet(\mathcal{A})$

"Pf": Take $\tau_{\leq n}$ for example and the other three are exercises.

If

$$\begin{tikzcd}
A^\bullet = \arrow[d, "f^\bullet"] & \cdots \arrow[r] & A^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"] & A^n \arrow[r, "d^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
B^\bullet = & \cdots \arrow[r] & B^{n-1} \arrow[r, "d^{n-1}"'] & B^n \arrow[r, "d^n"'] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

is a morphism of cochain complexes, then $d^n \circ f^n = f^{n+1} \circ d^n$

$\Rightarrow d^n \circ f^n(\ker d^n) = 0 \Rightarrow f^n(\ker d^n) \subseteq \ker d^n$.

Hence by restricting f^n to $\ker d^n = (\tau_{\leq n}(A^\bullet))^n$, we get a morphism of cochain complexes

$$\begin{tikzcd}
\tau_{\leq n} A^\bullet = \arrow[d, "\tau_{\leq n} f"] & \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, "f^{n-1}"] & \ker d^n \arrow[r] \arrow[d, "f^n"] & 0 \arrow[r] \arrow[d] & \cdots \\
\tau_{\leq n} B^\bullet = & \cdots \arrow[r] & B^{n-1} \arrow[r] & \ker d^n \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

$\square$

Exercise: Prove that $\tau_{\leq n} \circ \tau_{\geq n} = \tau_{\geq n} \circ \tau_{\leq n} = H^n$

Why are $\tau_{\leq n}, \tau_{\geq n}$ called canonical while $\sigma_{\geq n}, \sigma_{\leq n}$ are called stupid?

<u>Lemma:</u> ① $\tau_{\leq n}, \tau_{\geq n}$ induce functors $\tau_{\leq n}, \tau_{\geq n}: K(\mathcal{A}) \to K(\mathcal{A})$ (a)

and $\tau_{\leq n}, \tau_{\geq n}: D(\mathcal{A}) \to D(\mathcal{A})$. (b)

② $\sigma_{\leq n}, \sigma_{\geq n}$ cannot be defined on $K(\mathcal{A})$ or $D(\mathcal{A})$

<u>Idea of the proof</u> (details left as an exercise)

① For (a), need to prove that if $f^{\cdot}, g^{\cdot}: A^{\cdot} \to B^{\cdot}$ are homotopic, then $\tau_{\leq n}(f^{\cdot})$ and $\tau_{\leq}(g^{\cdot})$ are also homotopic.

For (b), need to prove that if $f^{\cdot}: A^{\cdot} \to B^{\cdot}$ is a quasiisomorphism, then so is $\tau_{\leq n}(f^{\cdot})$.

② Let $A^{\cdot} = \cdots \to 0 \to A \xrightarrow{d^{n-1}} \underset{\text{degree } n}{A \oplus B} \xrightarrow{d^n} B \to 0 \to \cdots$ $d^{n-1}(a) = (a, 0)$, $d^n(a,b) = b$

Then $Id_{A^{\cdot}}$ is homotopic to zero via

$$\begin{array}{l} A^{\cdot} = \cdots \to 0 \to A \to A \oplus B \to B \to 0 \to \cdots \\ \qquad\qquad\qquad \| \quad \swarrow h^n \quad \| \quad \swarrow h^{n+1} \quad \| \\ A^{\cdot} = \cdots \to 0 \to A \to A \oplus B \to B \to 0 \to \cdots \end{array}$$

where $h^{n+1}(b) = (0, b)$

$h^n(a,b) = a$

But $\sigma_{\leq n} A^{\cdot} = \cdots \to 0 \to A \to A \oplus B \to 0 \to \cdots$ with $H^n \cong B \neq 0$

$\sigma_{\geq n} A^{\cdot} = \cdots \to 0 \to A \oplus B \to B \to 0 \to \cdots$ with $H^n \cong A \neq 0$

Hence $\sigma_{\leq n}(Id_{A^{\cdot}})$ and $\sigma_{\geq n}(Id_{A^{\cdot}})$ are not homotopic to 0.

Properties of truncations (This is the prototype of a t-structure)

Let $D^{\geq n}(\mathcal{A})$ be the full subcategory of $D(\mathcal{A})$ consisting of all $A^\cdot$ s.t. $H^q(A^\cdot)=0$ $\forall q<n$, and $D^{\leq n}(\mathcal{A})$ be the full subcategory of $D(\mathcal{A})$ consisting of all $A^\cdot$ s.t. $H^q(A^\cdot)=0$ $\forall q>n$.

Then (a) $D^{\leq 0}(\mathcal{A})$, $D^{\geq 0}(\mathcal{A})$ are strictly full subcategories of $D(\mathcal{A})$

(b) $D^{\leq n}(\mathcal{A}) = \Sigma^{-n} D^{\leq 0}(\mathcal{A})$, $D^{\geq n}(\mathcal{A}) = \Sigma^{-n} D^{\geq 0}(\mathcal{A})$

and

(t1) $D^{\leq 0}(\mathcal{A}) \subseteq D^{\leq 1}(\mathcal{A})$, $D^{\geq 0}(\mathcal{A}) \supseteq D^{\geq 1}(\mathcal{A})$

(t2) $\mathrm{Hom}_{D(\mathcal{A})}(X,Y)=0$ for $X \in D^{\leq 0}(\mathcal{A})$ and $Y \in D^{\geq 1}(\mathcal{A})$

(t3) For any $X \in D(\mathcal{A})$, $\exists$ an exact triangle $A \to X \to B \to \Sigma A$ such that $A \in D^{\leq 0}(\mathcal{A})$ and $B \in D^{\geq 1}(\mathcal{A})$

Exercise $i: \tau_{\leq n} A \to A$ is quasiisom iff $A \in D^{\leq n}(\mathcal{A})$

$q: A \to \tau_{\geq n} A$ is quasiisom iff $A \in D^{\geq n}(\mathcal{A})$

Sketch of the proof: (t1) is directly by the definition.

For (t2), (t3), we can prove stronger versions

(t2)' If $m>n$, then $\mathrm{Hom}_{D(\mathcal{A})}(X,Y)=0$ for $X\in D^{\leq n}(\mathcal{A})$ and $Y\in D^{\geq m}(\mathcal{A})$

(t3)' For any $X\in D(\mathcal{A})$, $\exists$ an exact triangle

$$A\to X\to B\to \Sigma A$$ such that $A\in D^{\leq 0}(\mathcal{A})$ and $B\in D^{\geq 1}(\mathcal{A})$

(t2)': Since $Y\in D^{\geq m}(\mathcal{A})$, we have $Y\cong \tau_{\geq m}(Y)$

By replacing Y with $\tau_{\geq m}(Y)$, we can assume that $Y^q=0$ if $q<m$.

Given a roof
$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
with s a quasiisomorphism, we have $H^q(Z)=H^q(X)=0$ if $q>n$,

hence $\tau_{\leq n}Z \overset{i}{\xrightarrow{\sim}} Z$ in $D(\mathcal{A})$.

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & \tau_{\leq n}Z \arrow[u, "i"] \arrow[l] \arrow[r] & Y \\ & \tau_{\leq n}Z \arrow[u, equal] \arrow[ul, bend left] \arrow[ur, bend right] & \end{tikzcd}$$

Thus the roof
$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
is equivalent to the roof
$$\begin{tikzcd} & \tau_{\leq n}Z \arrow[dl, "s\circ i"'] \arrow[dr, "f\circ i"] & \\ X & & Y \end{tikzcd}$$

Now $f\circ i : \tau_{\le n} Z \to Y$ is a homotopy class of cochain maps from $\tau_{\le n} Z$ to Y with $(\tau_{\le n} Z)^q = 0$ when $q > n$ and $Y^q = 0$ when $q < m$, and $m > n$

$$\begin{tikzcd}
\tau_{\le n} Z = \arrow[d, "f\circ i"'] & \cdots \arrow[r] & Z^{n-1} \arrow[r] \arrow[d] & \ker d^n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
Y = & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & Y^m \arrow[r] & Y^{m+1} \arrow[r] & \cdots
\end{tikzcd}$$

Hence $f\circ i = 0$.

Exercise "Imitating" the above proof, prove that the natural functor $D : \mathcal{A} \longrightarrow D(\mathcal{A})$ is fully faithful.

$$A \longmapsto (\cdots \to 0 \to \underset{\text{degree } 0}{A} \to 0 \to \cdots)$$

Hence $\mathcal{A} \cong$ the essential image of D

$\cong D^{\le 0}(\mathcal{A}) \cap D^{\ge 0}(\mathcal{A})$ (Hint: use truncation again).

In particular, we have

$\boxed{D^{\le 0}(\mathcal{A}) \cap D^{\ge 0}(\mathcal{A}) \text{ is a full abelian subcategory of } D(\mathcal{A})}$

Proof of (t3)': It is a direct result of the following $\square$

Proposition: $\forall A^\bullet$ $\forall n \in \mathbb{Z}$, $\exists$ a unique morphism in $D(\mathcal{A})$

$h : \tau_{\geq n+1} A^\bullet \to \Sigma \tau_{\leq n} A^\bullet$ such that

$\tau_{\leq n} A^\bullet \xrightarrow{i} A^\bullet \xrightarrow{g} \tau_{\geq n+1} A^\bullet \xrightarrow{h} \Sigma \tau_{\leq n} A^\bullet$ is an exact triangle in $D(\mathcal{A})$

Idea of the proof

Step 1: Build exact triangles in $D(\mathcal{A})$ from SES's in $C(\mathcal{A})$.

More precisely, if $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ is an SES in $C(\mathcal{A})$

then $\exists$ (not necessarily unique) a morphism $Z^\bullet \to \Sigma X^\bullet$ in $D(\mathcal{A})$ (important! in general it doesn't exist in $K(\mathcal{A})$)

s.t. $X^\bullet \to Y^\bullet \to Z^\bullet \to \Sigma X^\bullet$ is an exact triangle in $D(\mathcal{A})$

Idea of the proof: $m: C(f)^\bullet \to Z^\bullet$ defined by

$$m^n : X^{n+1} \oplus Y^n \longrightarrow Z^n \quad \text{is a quasiisomorphism}$$
$$(x, y) \longmapsto g(y)$$

Hence
$$\begin{tikzcd} & C(f)^\bullet \arrow[dl, "m"'] \arrow[dr] & \\ Z^\bullet & & \Sigma X^\bullet \end{tikzcd}$$
defines a morphism in $D(\mathcal{A})$

Step 2: Consider the SES in $C(\mathcal{A})$

$$0 \to \tau_{\le n} A^\cdot \xrightarrow{i} A^\cdot \to Q^\cdot \to 0$$

By the definition of $\tau_{\le n}$, we have $Q^i = \begin{cases} 0 & i<n \\ A^n/\ker(d^n) & i=n \\ A^i & i>n \end{cases}$

and
$$\begin{tikzcd}
Q^\cdot = \arrow[d] & \cdots 0 \arrow[r] \arrow[d] & A^n/\ker(d^n) \arrow[r, "d^n"] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & A^{n+2} \arrow[r] \arrow[d] & \cdots \\
\tau_{\ge n+1} A^\cdot = & 0 \arrow[r] & 0 \arrow[r] & \operatorname{coker} d^{n-1} \arrow[r] & A^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$
is a quasiisomorphism

Hence $Q^\cdot \cong \tau_{\ge n+1} A^\cdot$ in $D(\mathcal{A})$ and by Step 1

$\exists$ morphism $h: \tau_{\ge n+1} A^\cdot \to \Sigma \tau_{\le n} A^\cdot$ in $D(\mathcal{A})$

s.t. $\tau_{\le n} A^\cdot \xrightarrow{i} A^\cdot \xrightarrow{q} \tau_{\ge n+1} A^\cdot \xrightarrow{h} \Sigma \tau_{\le n} A^\cdot$ is an exact triangle.

Step 3: It remains to prove the uniqueness of h.

It follows from the second lemma on page 12. $\square$

Furthermore, we have

Proposition: (1) $\tau_{\le n}: D(\mathcal{A}) \to D^{\le n}(\mathcal{A})$ is a right adjoint of the inclusion functor $D^{\le n}(\mathcal{A}) \to D(\mathcal{A})$

(2) $\tau_{\ge n}: D(\mathcal{A}) \to D^{\ge n}(\mathcal{A})$ is a left adjoint of the inclusion functor $D^{\ge n}(\mathcal{A}) \to D(\mathcal{A})$

Idea of the proof: (1) Need to prove: if $A \in D^{\le n}(\mathcal{A})$

then $\mathrm{Hom}_{D(\mathcal{A})}(A, \tau_{\le n} B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B)$

$$\varphi \longmapsto i \circ \varphi$$

(recall $i: \tau_{\le n} B \to B$)

Given a roof

$$\begin{tikzcd} & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

we can replace C by $\tau_{\le n} C$, . . . $\square$

• t-Structure and heart

<u>Definition:</u> Let $\mathcal{D}$ be a triangulated category. A t-structure on $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the following conditions:

If we put $\mathcal{D}^{\leq n} = \Sigma^{-n}(\mathcal{D}^{\leq 0})$ and $\mathcal{D}^{\geq n} = \Sigma^{-n}(\mathcal{D}^{\geq 0})$

then

(t1) $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$, $\mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$

(t2) $\operatorname{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$

(t3) For any $X \in \mathcal{D}$, $\exists$ an exact triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow \Sigma A$$ such that $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$

The heart of this t-Structure is $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$

Lemma: The Δ in (t3) is unique. More precisely,

if

$$\begin{tikzcd} A \arrow[r] & X \arrow[r] \arrow[d, equal] & B \arrow[r] & \Sigma A \\ A' \arrow[r] & X \arrow[r] & B' \arrow[r] & \Sigma A' \end{tikzcd}$$

are both exact with $A, A' \in \mathcal{D}^{\leq 0}$ and $B, B' \in \mathcal{D}^{\geq 1}$

then $\exists$ unique u, v s.t.

$$\begin{tikzcd} A \arrow[r] \arrow[d, dashed, "u"] & X \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, dashed, "v"] & \Sigma A \arrow[d, dashed, "\Sigma u"] \\ A' \arrow[r] & X \arrow[r] & B' \arrow[r] & \Sigma A' \end{tikzcd}$$

commutes.

Furthermore, they give an isomorphism of Δ's.

Idea of the proof: It follows from the following lemma.

Lemma: Given

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] & \Sigma A \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma A' \end{tikzcd}$$

→ exact

→ exact

If $g' \circ v \circ f = 0$, then $\exists$ u, w such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, dashed, "w"] & \Sigma A \arrow[d, dashed, "\Sigma u"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma A' \end{tikzcd}$$

commutes.

If furthermore $\mathrm{Hom}_{\mathcal{D}}(X, Z'[-1]) = 0$, then u and w are unique.

Proof as an exercise (Hint, $\mathrm{Hom}_{\mathcal{D}}(\ \ )$ induces long exact sequences)

$\square$

Proposition ① $\exists\ \tau_{\leq n}: \mathcal{D} \to \mathcal{D}^{\leq n}$ a right adjoint of the inclusion functor

② $\exists\ \tau_{\geq n}: \mathcal{D} \to \mathcal{D}^{\geq n}$ a left adjoint of the inclusion functor

Idea of the proof: $\forall\ X \in \mathcal{D}$, $\forall\ n \in \mathbb{Z}$, from (t3) $\exists!$ $A \to X \to B \to \Sigma A$ exact. Define $\tau_{\leq n} X = A$ and $\tau_{\geq n+1} X = B$. $\boxtimes$

Theorem. The heart $\mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0} = \mathcal{A}$ is a full abelian subcategory of $\mathcal{D}$

- Furthermore, for $A, B, C \in \mathcal{A}$, $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact in $\mathcal{A}$ if and only if $\exists$ $C \xrightarrow{h} \Sigma A$ in $\mathcal{D}$ st.

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$ is an exact triangle in $\mathcal{D}$.

Idea of the proof: The second statement is actually how we prove that $\mathcal{A}$ is abelian, i.e. we find kernels and cokernels by exact triangles. $\boxtimes$

Exercise: proof the second statement directly for the natural t-structure on $D(\mathcal{A})$.

Proposition: $\tau_{\leq 0} \circ \tau_{\geq 0} = \tau_{\geq 0} \circ \tau_{\leq 0} : \mathcal{D} \to \mathcal{A}$

is a cohomological functor, denote by ${}^tH^0$.

- t-exact functors

Def: Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories endowed with t-structures $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$.

A triangulated functor $F: \mathcal{D} \to \mathcal{D}'$ is called

- right exact if $F(\mathcal{D}^{\leq 0}) \subseteq \mathcal{D}'^{\leq 0}$
- left exact if $F(\mathcal{D}^{\geq 0}) \subseteq \mathcal{D}'^{\geq 0}$
- exact if both right exact and left exact.

- Some comments

- Derived equivalence: There exist abelian categories $\mathcal{A} \not\cong \mathcal{B}$ with $D(\mathcal{A}) = D(\mathcal{B}) = \mathcal{D}$ (this is called "derived equivalence").

  This implies that $\mathcal{D}$ admits two non-equivalent t-structures, one with heart $\mathcal{A}$ and the other with heart $\mathcal{B}$.

  A such example is where $\mathcal{A} = \mathrm{Coh}(\mathbb{P}^1(k))$ k a field

  $\mathcal{B} =$ k-reps of the quiver $\bullet \rightrightarrows \bullet$

- In all of the discussions, we can replace $D(\mathcal{A})$ by

  $D^b(\mathcal{A})$, $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ where

$D^b(\mathcal{A})$ = full subcategory of $D(\mathcal{A})$ consisting of $A^\bullet$ s.t. $A^q = 0$ when $|q| >> 0$

$D^+(\mathcal{A})$ = full subcategory of $D(\mathcal{A})$ consisting of $A^\bullet$ s.t. $A^q = 0$ when $q << 0$

$D^-(\mathcal{A})$ = full subcategory of $D(\mathcal{A})$ consisting of $A^\bullet$ s.t. $A^q = 0$ when $q >> 0$

Alternatively, we can also construct $D^b(\mathcal{A})$, $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ from $C^b(\mathcal{A})$, $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ (bounded (below/above) cochains).

Exercise: Use truncation functors to prove that the two constructions are the same

- Warning: Let D be a triangulated category $(D^{\le 0}, D^{\ge 0})$ be a t-structure, and $\mathcal{A} = D^{\le 0} \cap D^{\ge 0}$ be the heart.

In general it is not true that $D(\mathcal{A}) \simeq D$!

Even the existence of a functor $D(\mathcal{A}) \to D$ that preserves $\mathcal{A}$ is not guaranteed. (If exist, it's called a realization functor, see Achar's note Thm A.7.16)

Stupid example: $D = D^b(\mathcal{A})$ (instead of $D(\mathcal{A})$)).