Sheaves, Local systems and their derived functors.

References:

- Achar: Introduction to perverse sheaves.

Assumptions:

- X, Y are topological spaces that are compact, Hausdorff, second countable, locally path-connected, semi-locally simply-connected, and path connected.
- All sheaves are sheaves of $\mathbb{C}$-vector spaces.

Operations on Sheaves:

Direct Sums: $\mathcal{F}, \mathcal{G} \in Sh(X)$,

$\mathcal{F} \oplus \mathcal{G} \in Sh(X)$ with sections:

$$(\mathcal{F} \oplus \mathcal{G})(U) := \mathcal{F}(U) \oplus \mathcal{G}(U).$$

Push forward: $\mathcal{F} \in Sh(X)$, $f : X \to Y$ cts,

$f_* \mathcal{F} \in Sh(Y)$ with sections:

$(f_* \mathcal{F})(U) := \mathcal{F}(f^{-1}(U)) \quad U \subset Y.$

Pull Back: $\mathcal{F} \in Sh(Y)$, $f : X \to Y$ cts.,

$f^{-1}\mathcal{F} := \left(ps\, f^{-1} \mathcal{F}\right)^+$

where $\left(ps\, f^{-1}\mathcal{F}\right)$ has sections

$$\left(ps\, f^{-1}\mathcal{F}\right)(U) := \varinjlim_{V \supset f(U)} \mathcal{F}(V)$$

$$= \left\{ (V, s) \;\middle|\; \begin{smallmatrix} V \supset f(U) \\ s \in \mathcal{F}(V) \end{smallmatrix} \right\} / \sim$$

$(V, s) \sim (V', s')$ if $\exists\, W \overset{\circ}{\subset} V \cap V'$ s.t. $s|_W = s'|_W$.

Sheaf Hom: $\mathcal{F}, \mathcal{G} \in Sh(X)$. then:

$\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ is the sheaf with sections:

$$\mathcal{H}om_X(\mathcal{F}, \mathcal{G})(U) \neq \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U))$$
$$:= \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Tensor Product: $\mathcal{F}, \mathcal{G} \in Sh(X)$ then

$$\mathcal{F} \otimes \mathcal{G} := (\mathcal{F} \otimes_{ps} \mathcal{G})^+ \quad \text{where}$$

$(\mathcal{F} \otimes_{ps} \mathcal{G})$ has sections:

$$(\mathcal{F} \otimes_{ps} \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U).$$

Examples: (1) $f : X \to pt, \quad \mathcal{F} \in Sh(X)$

$$f_* \mathcal{F} \simeq \Gamma(X, \mathcal{F}).$$

(2) $i : pt \hookrightarrow X \quad \mathbb{C}^n \in Sh(pt).$

$i_* \mathcal{F}$ is a sky scraper sheaf.

i.e. $\quad i_* \mathcal{F}_x \simeq \begin{cases} \mathbb{C}^n & \text{if } x = x_{pt} \\ 0 & \text{if } x \neq x_{pt}. \end{cases}$

(3) $f : \mathbb{C}^* \longrightarrow \mathbb{C}^* \qquad z \longmapsto z^2$

$$f_* \underline{\mathbb{C}} \simeq \underline{\mathbb{C}} \oplus Q$$

where $Q(U) = \left\{ g(z) : U \to \mathbb{C}^* \;\middle|\; 2z \frac{dg}{dz} = g \right\}.$

$g(z) = e^{\frac{1}{2}\ln(z)} = e^{\ln(\sqrt{z})}.$

(4) $f : X \to Y,$

$$f^{-1} \underline{\mathbb{C}}^n_Y \simeq \underline{\mathbb{C}}_X.$$

(5) $i : X \hookrightarrow Y, \qquad \mathcal{F} \in Sh(Y)$

$$f^{-1}\mathcal{F} \simeq \mathcal{F}|_X.$$

Adjoint Pairs: $\mathcal{A}, \mathcal{B}$ abelian categories,

$\left.\begin{array}{l} S : \mathcal{A} \longrightarrow \mathcal{B} \\ T : \mathcal{B} \longrightarrow \mathcal{A} \end{array}\right.$ functors

Then (S, T) form an adjoint pair

if:

$$\mathrm{Hom}_{\mathcal{B}}(S(A), B) \simeq \mathrm{Hom}_{\mathcal{A}}(A, T(B)).$$

+ natural in $\mathcal{A}$ and $\mathcal{B}$.

Thm: (f^{-1}, f_*) and $(- \otimes \mathcal{G}, \mathcal{H}om(\mathcal{G}, -))$.

are adjoint pairs for both

Hom and $\mathcal{H}om$.

Local Systems & constructible sheaves

Local System: $\mathcal{F} \in Sh(X)$ is a local system if

$\forall x \in X$, $\exists U \ni x$, s.t.

$\mathcal{F}|_U$ is a constant sheaf.

Non-Examples: (1) $X = \mathbb{C}$, $\mathcal{F}$ skyscraper sheaf.

(2) $X = \mathbb{C}$, $\mathcal{F}$ cts functions on X.

Examples: (1) $X = \mathbb{C}$, $\mathcal{F} = \underline{\mathbb{C}}^n_X$

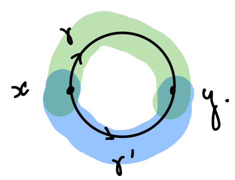
(2) $X = \mathbb{C}^*$, $\mathcal{F} \simeq \mathcal{Q}$.

Constructible Sheaves: $X = \coprod_{\lambda \in \Lambda} X_\lambda$ $|\Lambda| < \infty$, X_λ locally closed.

$\mathcal{F}$ is constructible if

$\mathcal{F}|_{X_x}$ is a local system.

Classification Theorem: $\mathcal{L}oc(X) \xrightarrow{\sim} \pi_1(X, x_0)$ - mod.



$\mathcal{L}_x \overset{\rho(\gamma)}{\simeq} \mathcal{L}_y$.

$\mathcal{L}_x \overset{\rho(\gamma')}{\simeq} \mathcal{L}_y$

$\rho(\gamma')^{-1}\rho(\gamma)$ is an auto of $\mathcal{L}_x$.

Proof Sketch:

Step 1: If $x_1, \dots, x_n \in K \subset X$, where K connected, open, $\exists V \subset K$ s.t. $\mathcal{F}|_V$ is locally constant.

then $\mathcal{F}_{x_i} \simeq \mathcal{F}_{x_j}$.

$\mathcal{F}|_V \simeq \mathcal{F}_{x_i}$.

Step 2: Paths & Path-homotopies can be partitioned into intervals / rectangles that satisfy the above conditions.

Step 3: A path $\gamma : [0,1] \to X$ defines $\rho(\gamma) : \mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$.

Step 4: $\gamma \sim \gamma' \Rightarrow \rho(\gamma) = \rho(\gamma')$.

Vantage. $\rho : \pi_1(X, x_0) \longrightarrow GL(\mathcal{F}_{x_0})$

Point :

Aim : Obtain an inverse.

Step 5: Show that the sheaf:

$$\mathcal{G}(U) := \{ k : U \to \mathbb{C}^n \mid \text{for any } \gamma : [0,1] \to U$$
$$k(\gamma(1)) = [\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}] \cdot k(\gamma(0)).$$

choice of fixed path.

is locally constant.

Step 6: These process are inverse to each other.

Derived Functors in the Category of Sheaves.

Aim: Introduce the adjoint pairs:

$$(\otimes^L, R\mathcal{H}om), \quad (f^{-1}, Rf_*), \quad (Rf_!, \underbrace{f^!}_{\text{subtle.}}).$$

Recall:

Graded Hom: $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in ch(sh(X))$, then

$$\underline{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) := \bigoplus_{k-j=i} Hom(\mathcal{F}^j, \mathcal{G}^k).$$

+ ugly differential.

Graded Tensor: $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in ch(sh(X))$, then

$$(\mathcal{F}^\bullet \underline{\otimes} \mathcal{G}^\bullet) := \bigoplus_{j+k=i} (\mathcal{F}^j \otimes \mathcal{G}^k).$$

+ ugly differential.

Now we introduce the derived analogues:

Right Derived Sheaf Hom : $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D(X)$, $I^\bullet$ an injective res. of $\mathcal{G}$.

$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) := \underline{\mathcal{H}om}(\mathcal{F}^\bullet, I^\bullet).$$

Left Derived Tensor Product : $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D(X)$, $P^\bullet$ a flat res. of $\mathcal{G}$

$$\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet := \mathcal{F}^\bullet \underline{\otimes} P^\bullet$$

Since $\mathcal{G}$ is a sheaf of $\mathbb{C}$-v.sp. $\mathcal{G}^\bullet = P^\bullet$.

Next we consider the proper push forward:

Proper Push-Forward : $f : X \to Y$ cts.

$f_! \mathcal{F}$ is the sheaf with sections:

$$(f_! \mathcal{F})(U) = \{ s \in \mathcal{F}(f^{-1}(U)) \mid f|_{\operatorname{supp} s} : \operatorname{supp} s \to U \text{ is proper} \}.$$

What are our adapted classes?

Functor	*Exactness*	*Adapted classes*	*Derived functor*	*Classical derived functors*
Γ	left	injective, flabby	$R\Gamma$	$H^i(X,-)$
f_*	left	injective, flabby	Rf_*	$R^i f_*$
Hom	left	injective	$R\,\mathrm{Hom}$	Ext^i
$\mathcal{H}om$	left	injective	$R\,\mathcal{H}om$	$\mathcal{E}xt^i$
$f_!$	left	injective, flabby, soft	$Rf_!$	$R^i f_!$
f^{-1}	exact	—	f^{-1}	
$\otimes$	right	flat	$\otimes^L$	$\mathcal{T}or_i$

where flabby : $\mathcal{F}(X) \twoheadrightarrow \mathcal{F}(U) \quad \forall\, U \subset X$

$\Downarrow$

soft : $\Gamma(\mathcal{F}) \twoheadrightarrow \Gamma(\mathcal{F}|_K) \quad \forall\, K \subset X$ compact.

How do these operations behave w.r.t. their adapted classes?

f_* (injective / flabby) = injective / flabby.

$f_!$ (soft) = soft.

$\mathcal{H}om$ (flat / anything, inj) = flabby / injective.

Adjunction Properties: $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^-(X)$, $\mathcal{H}^\bullet \in D^+(X)$

$$\mathrm{Hom}(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq \mathrm{Hom}(\mathcal{F}^\bullet, R\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet))$$

$$R\mathcal{H}om(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq R\mathcal{H}om(\mathcal{F}^\bullet, R\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet))$$

$R\Gamma$

$$R\mathrm{Hom}(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq R\mathrm{Hom}(\mathcal{F}^\bullet, R\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet))$$

$f : X \to Y$ cts. $\mathcal{F}^\bullet \in D^-(Y)$, $\mathcal{G}^\bullet \in D^+(X)$

$$\mathrm{Hom}(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathrm{Hom}(\mathcal{F}^\bullet, Rf_* \mathcal{G}^\bullet)$$

$$R\mathrm{Hom}(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\mathrm{Hom}(\mathcal{F}^\bullet, Rf_* \mathcal{G}^\bullet).$$

$$Rf_* R\mathcal{H}om(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\mathcal{H}om(\mathcal{F}^\bullet, Rf_* \mathcal{G}^\bullet).$$