# Wednesday Talk outline (Pramod notes)

Plan:

- Recall definition of $f_!$ (sec 1.3)
- Do Lemma 1.3.1 ($f_!$ exact for locally closed f; ~~stalks~~ stalks are same as for $f_*$, & $f_! = f_*$ if f is closed embedding)
- Define ${}^0f^!$ for locally closed embeddings (1.3.2)
- State Lemma 1.3.3 for open inclusions
- Prove that ${}^0f^!$ is left exact for locally-closed
- Explain that $\boxed{{}^0f^!}$ can be extended to $D^+$ by resolution & is right adjoint to $\boxed{f_!}$

Remark that this adjunction holds more generally & that adj

- State and prove Thm 1.3.10

- A.7. general recollement

- State general Verdier duality.

## Plan for today:

- Will define $f^!$
- Recollement or "glueing theory"
- Verdier duality

(Will mostly follow Achar 1.3, A.7, 2.8)

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## (I) Proper pushforward & pull back (for locally closed)

- Assume $h : Y \hookrightarrow X$ is a locally-closed inclusion (i.e. $\exists$ open subset $W \subset X$, s.t. $Y = W \cap \bar{Y}$)

### Proper forward "extension by zero"

To define this, we first define a functor

$${}^{\circ}h_! : Sh(Y, \Bbbk) \to Sh(X, \Bbbk)$$

where for $\mathcal{F} \in Sh(Y, \Bbbk)$,

${}^{\circ}h_!(\mathcal{F}) \in Sh(X, \Bbbk)$ is sheafification of presheaf ${}^{\circ}h_{!,pre}(\mathcal{F})$ given by:

$${}^{\circ}h_{!,pre}(\mathcal{F})(U) = \begin{cases} \mathcal{F}(U \cap Y) & \text{if } U \cap \bar{Y} \subset Y \\ 0 & \text{otherwise} \end{cases}$$

Facts: (1) ${}^{\circ}h_!$ is exact

(2) $${}^{\circ}h_!(\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

* Since ${}^{\circ}h_!$ is exact can automatically extend to

$$\boxed{h_! : D^+(Y, \mathbb{K}) \to D^+(X, \mathbb{K})}$$

**Exercise:** Show this is consistent with general definition of proper pushforward

**Fact:** If $i : Z \hookrightarrow X$ is closed, then

$$i_!(\mathcal{F}) \simeq i_*(\mathcal{F}) \quad \forall \quad \mathcal{F} \in D^+(Z, \mathbb{K})$$

**Proper pull back** "restriction with supports"

Define ${}^{\circ}h^! : Sh(X, \mathbb{K}) \to Sh(Y, \mathbb{K})$

$${}^{\circ}h^!(\mathcal{F})(U) = \varinjlim_{\substack{V \subset X \text{ open} \\ V \cap Y = U}} \{ s \in \mathcal{F}(V) \mid supp(s) \subset U \}$$

**Exercise:** Show that ${}^{\circ}h^!(\mathcal{F})$ is already a sheaf!

Note: $h^*(\mathcal{F})(U) = \varinjlim_{\substack{V \subset X \text{ open} \\ V \cap Y = U}} \mathcal{F}(V)$ (usual inverse image functor)

**Fact:** The functor $h^!(\mathcal{F}) : Sh(X, \mathbb{K}) \to Sh(Y, \mathbb{K})$ is left-exact. (for $h : Y \hookrightarrow X$ locally-closed)

$\rightsquigarrow$ set $\boxed{h^! := R^0 h^! : D^+(X, \mathbb{K}) \to D^+(Y, \mathbb{K})}$

**Fact:** If $j : U \hookrightarrow X$ is open, then
$j^! \mathcal{F} \simeq j^* \mathcal{F} \quad \forall\, \mathcal{F} \in D^+(X, \mathbb{K})$

**Fact:** $W \overset{k}{\hookrightarrow} Y \overset{h}{\hookrightarrow} X$ loc. closed

- $h_! k_! \mathcal{F} \simeq (h \circ k)_! \mathcal{F}$ for $\mathcal{F} \in D^+(W, \mathbb{K})$
- $k^! h^! \mathcal{G} \simeq (h \circ k)^! \mathcal{G}$ for $\mathcal{G} \in D^+(X, \mathbb{K})$

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**Theorem:** $h^!$ is the right adjoint of $h_!$ (for $h : Y \hookrightarrow X$ loc. closed). Explicitly, $\exists$ natural isom.

$$\boxed{\mathrm{Hom}_{D^+(X,\mathbb{K})}(h_! \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{D^+(Y,\mathbb{K})}(\mathcal{F}, h^! \mathcal{G})}$$

$\forall\, \mathcal{F} \in D^+(Y, \mathbb{K}),\quad \mathcal{G} \in D^+(X, \mathbb{K})$

Note: The right adjoint $f^!$ exists more generally for maps $f : X \to Y$ with "nice" homological properties

## Existence of $f^!$ for more general maps

Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian cats.

$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ the right derived funct

Then $F$ has cohomological dimension $\leq d$ if $\forall X \in \mathcal{A}$, $\boxed{RF(X) \in D^+(\mathcal{B})^{\leq d}}$

i.e. $R^n F(X) = H^n(RF X) = 0 \quad \forall n > d$

Now suppose $f : X \to Y$ is a map of locally compact spaces.

So ${}^0 f_! : Sh(X, \mathbb{K}) \to Sh(Y, \mathbb{K})$ exists

and we set:

$$\boxed{f_! := R\,{}^0 f_! : D^+(X, \mathbb{K}) \to D^+(Y, \mathbb{K})}$$

**Theorem** (Theorem 1.5.4)

If $f_!$ has finite cohomological dimension then $f_! : D^-(X, \mathbb{K}) \to D^-(Y, \mathbb{K})$ makes sense and right adjoint

$$f^! : D^+(Y, \mathbb{K}) \to D^+(X, \mathbb{K})$$

exists.

## (II) Recollement:

Fix $Z \overset{i}{\underset{\text{closed}}{\hookrightarrow}} X \overset{j}{\underset{\text{open}}{\hookleftarrow}} U$

with $\boxed{U = X \setminus Z}$

Goal: Understand $D^+(X, \mathbb{K})$ in terms of $D^+(Z, \mathbb{K})$ & $D^+(U, \mathbb{K})$ via "gluing theory"

"Recollement (gluing) diagram":

left-adjoints

$$
\begin{tikzcd}
D^+(Z, \mathbb{K}) \arrow[r, "i_* = i_!"] & D^+(X, \mathbb{K}) \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^! = j^*"] & D^+(U, \mathbb{K}) \arrow[l, bend right=60, "j_*"'] \arrow[l, bend left=60, "j_!"]
\end{tikzcd}
$$

right adjoints

Theorem (Gluing theorem) Achar Thm 1.3.10

(1) $i^* \circ j_! = i^! \circ j_* = j^* \circ i_* = 0$

(2) $\forall \mathcal{F} \in D^+(X, \mathbb{k})$ $\exists$ dist. $\Delta$.

(*) $$j_! j^* \mathcal{F} \xrightarrow[\text{for } (j_!, j^*)]{\text{counit}} \mathcal{F} \xrightarrow[\text{for } (i^*, i_*)]{\text{unit}} i_* i^* \mathcal{F} \to$$

(**) $$i_* i^! \mathcal{F} \xrightarrow{\text{counit}} \mathcal{F} \xrightarrow{\text{unit}} j_* j^* \mathcal{F} \to$$

(3) If $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ any $\Delta$ with $i^* \mathcal{F}' = j^* \mathcal{F}'' = 0$, then can. iso. to (*)

(4) Similar statement for (**)

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Proof (1): can verify by checking on level of abelian categories (exercise)

(2) Will construct (*).

First suppose $\mathcal{F} \in Sh(X, \mathbb{k})$. Then $j_! j^* \mathcal{F}$ & $i_* i^* \mathcal{F}$ are objects of $Sh(X, \mathbb{k})$ & we have a sequence:

$$(\dagger) \quad 0 \to j_! j^* \mathcal{F} \xrightarrow{\text{counit}} \mathcal{F} \xrightarrow{\text{unit}} i_* i^* \mathcal{F} \to 0$$

Claim: This sequence is exact

Proof: Suffices to show that $\forall x \in X$,

$$0 \to (j_! j^* \mathcal{F})_x \xrightarrow{\delta} \mathcal{F}_x \xrightarrow{\beta} (i_* i^* \mathcal{F})_x \to 0$$

is exact.

- If $x \in U$, then $\delta$ is an iso & $\beta = 0$ $\Rightarrow$ exact
- If $x \in Z := X \setminus U$, then $\delta = 0$ & $\beta$ is an iso $\Rightarrow$ exact

$\Rightarrow$ the sequence is exact

Now let $\mathcal{F} \in D^+(X, \mathbb{k})$. (So $\mathcal{F}$ is represented by a chain complex).

$(\dagger)$ gives a SES of chain complexes

By standard homological algebra arguments

$\rightsquigarrow$ $(*)$ is a distinguished triangle

[ $(**)$ can be deduced from $(*)$ via Verdier duality ]

(3) See Pramod's book $\square$

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Corollary:

(1) $i_* : D^+(Z, \mathbb{K}) \to D^+(X, \mathbb{K})$ is fully faithful

& induces $D^+(Z, \mathbb{K}) \xrightarrow{\sim} \{ \mathcal{F} \in D^+(X, \mathbb{K}) \mid \text{supp}\, \mathcal{F} \subset Z \}$

(2) $\forall \mathcal{F} \in D^+(U, \mathbb{K})$, $i^* j_* \mathcal{F} \simeq i^! j_! \mathcal{F}[1]$

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Application to t-structures:

Let $\mathcal{D}' = D_c^b(Z, \mathbb{K})$, $\mathcal{D} = D_c^b(X, \mathbb{K})$

$\mathcal{D}'' = D_c^b(U, \mathbb{K})$

Suppose $\mathcal{D}'$ & $\mathcal{D}''$ have t-structures

$(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$, $(\mathcal{D}''^{\leq 0}, \mathcal{D}''^{\geq 0})$,

then $\exists$ t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$

given by

- $\mathcal{D}^{\le 0} = \{ \mathcal{F} \in \mathcal{D}^b_c(X, \mathbb{K}) \mid i^* \mathcal{F} \in \mathcal{D}'^{\le 0}, \; j^* \mathcal{F} \in \mathcal{D}''^{\le 0} \}$

- $\mathcal{D}^{\ge 0} = \{ \mathcal{F} \in \mathcal{D}^b_c(X, \mathbb{K}) \mid i^! \mathcal{F} \in \mathcal{D}'^{\ge 0}, \; j^* \mathcal{F} \in \mathcal{D}''^{\ge 0} \}$

This is the unique t-structure where $i_*$ & $j^*$ are t-exact.

(See Exercise A.7.6 in Pramod's notes)

- Remark: This method of gluing t-structures leads to definition of perverse sheaves.

In general (see Exercise A.7.4 in Pramod's book) A recollement diagram consists of three $\Delta$-cats & six functors:

$$
\begin{tikzcd}
\mathcal{T}' \arrow[r, "\iota"] & \mathcal{T} \arrow[l, bend right, "\iota^L"'] \arrow[l, bend left, "\iota^R"] \arrow[r, "\pi"] & \mathcal{T}'' \arrow[l, bend right, "\pi^L"'] \arrow[l, bend left, "\pi^R"]
\end{tikzcd}
$$

where functors satisfy:

(a) $(i^L, i)$ $(i, i^R)$ are adjoint pairs

(b) $(\pi^L, \pi)$, $(\pi, \pi^R)$ are adjoint pairs

(c) $\pi \circ \iota = 0$

(d) $\forall X \in \mathcal{T}$ $\exists$ $\Delta$-triangles

$$\iota\iota^R(X) \to X \to \pi^R\pi(X) \to$$
$$\pi^L\pi(X) \to X \to \iota\iota^L(X) \to$$

(e) $\iota$, $\pi^L$ & $\pi^R$ are fully-faithful.

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Allows us to build t-structures $\mathcal{T}$ by gluing t-structures on $\mathcal{T}'$ & $\mathcal{T}''$.

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## [II] Verdier Duality

Let $X$ be locally compact & $a_X : X \to pt$ the constant map.

We say that $X$ has <u>c-soft dimension</u> $\leq n$ if $a_{X!}$ has cohom. dim. $\leq n$.

Assume from now on that $X$ has $c$-soft dimension $\leq n$.

Let $\omega_X := a_X^! \mathbb{K}_{pt} \in D^+(X, \mathbb{K})$ be the dualizing complex of $X$.

The Verdier duality functor

$\mathbb{D} : D^-(X, \mathbb{K})^{op} \longrightarrow D^+(X, \mathbb{K})$ is given by

$\mathbb{D}(\mathcal{F}) = R\mathcal{H}om(\mathcal{F}, \omega_X)$

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Now assume $f : X \to Y$ is such that $f_!$ has finite cohom. dimension.

Properties:

(1) $\exists$ canonical isomorphism $f^! \omega_Y \simeq \omega_X$

(2) $f_* \mathbb{D}(\mathcal{F}) \simeq \mathbb{D}(f_! \mathcal{F}) \quad \forall \mathcal{F} \in D^+(X, \mathbb{K})$

(3) $f^! \mathbb{D}(\mathcal{G}) \simeq \mathbb{D}(f^* \mathcal{G}) \quad \forall \mathcal{G} \in D^-(Y, \mathbb{K})$

Poincare duality

Thm If $X$ is a compact orientable $n$-dimensional manifold, then

$$\boxed{\omega_X = \mathbb{K}_X\{n\}}$$

Note that singular cohomology can be given by

$$H^i(X,\mathbb{K}) := H^i(a_{X*}\mathbb{K}_X) \qquad \forall i$$

$$= a_{X*}\mathcal{H}om^i(\mathbb{K}_X,\mathbb{K}_X) = \mathrm{Hom}_{D^+(X,\mathbb{K})}(\mathbb{K}_X,\mathbb{K}_X\{i\})$$

Compactly supported cohomology is defined as

$$\boxed{H^i_c(X,\mathbb{K}) := H^i(a_{X!}\mathbb{K}_X) \qquad \forall i.}$$

Property (2) implies that

$$a_{X*}(\mathbb{D}_X\mathbb{K}_X) \simeq \mathbb{D}_{pt}(a_{X!}\mathbb{K}_X)$$

$$\begin{array}{ccc} a_{X*}(\mathbb{D}_X\mathbb{K}_X) & \simeq & \mathbb{D}_{pt}(a_{X!}\mathbb{K}_X) \\ \| & & \wr \\ a_{X*}(R\mathcal{H}om(\mathbb{K}_X,\mathbb{K}_X\{n\})) & \simeq & R\mathcal{H}om(a_{X!}\mathbb{K}_X,\mathbb{K}_{pt}) \\ \wr & & \\ R\mathrm{Hom}(\mathbb{K}_X,\mathbb{K}_X\{n\}) & & \end{array}$$

So $R\mathrm{Hom}(\mathbb{K}_X,\mathbb{K}_X\{n\}) \simeq R\mathcal{H}om(a_{X!}\mathbb{K}_X,\mathbb{K}_{pt})$

Take $H^{-i}(-)$ of both sides:

$$\boxed{\begin{aligned} H^{-i}(R\mathcal{H}om(\mathbb{K}_X, \mathbb{K}_X\{n\})) &= \mathrm{Hom}(\mathbb{K}_X, \mathbb{K}_X\{n-i\}) \\ &= H^{n-i}(X, \mathbb{K}) \end{aligned}}$$

While $H^{-i}(RHS)$ obtained by resolving

$$\mathbb{K}_X \to I_X^\bullet : I_X^0 \to I_X^1 \to \cdots$$

by injectives

& apply $\mathrm{Hom}^{-i}(a_{X!} I_X^\bullet, \mathbb{K}) =$

$$H^{-i}\left((a_{X!} I_X^0)^* \leftarrow (a_{X!} I_X^1)^* \leftarrow (a_{X!} I_X^2)^* \leftarrow \cdots\right)$$

$$= \boxed{H^i_c(X, \mathbb{K})^*}$$