Def: X a variety over $\mathbb{C}$, a <u>stratification</u> S of X is a decomposition into smooth locally closed subvarieties (called strata)

$$X = \bigsqcup_{s \in S} X_s$$

Such that each $\overline{X_s}$ is a union of strata.

Example: $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \sqcup pt \cong \mathbb{A}^1 \sqcup \mathbb{A}^0$

$$GL_n(\mathbb{C}) = \bigsqcup_{w \in S_n} BwB \qquad \text{Bruhat decomposition.}$$

Remark:

Moving away from varieties, one needs to impose techical conditions for a well behaved theory, ~~the main~~ for istance, each point ~~local~~ has a neighbourhood N with $N \cong \mathbb{R}^k \times CL$.

Def: A sheaf $\mathcal{F}$ on a stratified space (X, S) is <u>S-constructible</u> if for each inclusion $i_s : X_s \hookrightarrow X$ we have $i_s^* \mathcal{F}$ is locally constant on X_s.

A complex of sheaves $\mathcal{F}$ is S-constructible if its cohomology sheaves $H^i(\mathcal{F})$ are S-constructible.

A sheaf (or complex of sheaves) is just <u>constructible</u> if it is S-constructible for some stratification S.

Example: On $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \sqcup pt$

a constructible sheaf wrt this stratification

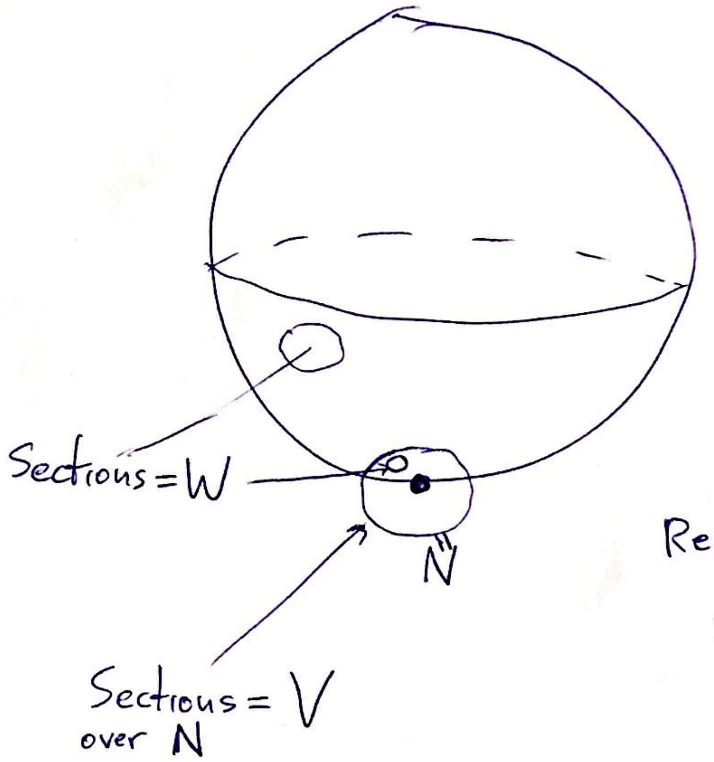
is the data of $V \xrightarrow{\mathcal{F}} W$

a map of vector spaces.

V = the stalk at the point $= \mathcal{F}(N)$

W = the stalk at any other point.

Picture:



Restriction map is

$$\begin{array}{ccc} \Gamma(N, \mathcal{F}) & \longrightarrow & \Gamma(N - pt, \mathcal{F}) \\ \| & & \| \\ V & & W \end{array}$$

Def: $D^b_c(X)$ (or $D^b_{c,S}(X)$) is the full subcategory of $D^b(X)$ spanned by constructible complexes. (or S constructible)

~~Com~~. Remark: This is <u>not</u> the derived category of S constructible sheaves (when we fix S).

· Exercise: Prove this, take X to be any simply connected manifold of dimension ≥ 2 with the trivial stratification.

Theorem: For a map $X \xrightarrow{f} Y$ the functors $f_*, f_!, f^*, f^!, \otimes, \mathcal{H}om$ on derived categories preserve constructibility.

For nice maps, we can sometimes preserve constructiblity wrt fixed stratifications.

(See Achar chapter 2)

t structures:

From $D^b(X)$, we get the standard t structure on $D^b_{c,S}(X)$.

Observe that for $U \overset{j}{\hookrightarrow} X$ an open stratum, we have j^* is t-exact, as is i_* for $Z \overset{i}{\hookrightarrow} X$ its complement.

Thus, by the recollement result last time, (and induction) this t-structure is the gluing of the standard t structures on the strata:

$$D^b_{c,S}(X_S) = \text{Sheaves with locally constant cohomology.}$$

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Perverse t structure:

We can simply describe this t structure as the gluing of the standard one on each strata, shifted down by $\dim X_s$. (Where $\dim X_s$ is the complex dimension, half the real dimension).

Why might this be a good idea?

This t structure is preserved by Verdier duality on each strata, and since j^*, i_* commute with $\mathbb{D}$, it is preserved by $\mathbb{D}$ on X.

We will now make this more explicit.

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We will treat the open inclusion case,

$$U \xhookrightarrow[j]{\circ} X \xhookleftarrow[i]{} Z$$

This t structure can be described inductively by

$$ {}^pD_X^{\leq 0} = \left\{ \mathcal{F} \text{ with } \begin{array}{l} j^*\mathcal{F} \in D^{\leq -\dim U} \\ i^*\mathcal{F} \in {}^pD_Z^{\leq 0} \end{array} \right\} $$

$$ {}^pD_X^{\geq 0} = \left\{ \mathcal{F} \text{ with } \begin{array}{l} j^*\mathcal{F} \in D^{\geq -\dim U} \\ i^!\mathcal{F} \in {}^pD_Z^{\geq 0} \end{array} \right\} $$

So unwinding this inductive definition, we arrive at:

$$ {}^pD_X^{\cancel{\geq}\leq 0} = \left\{ \begin{array}{l} \mathcal{F} \text{ with } i_S^*\mathcal{F} \in D_S^{\leq -\dim S} \\ \text{for all strata } S \end{array} \right\} $$

$$ {}^pD_X^{\geq 0} = \left\{ \begin{array}{l} \mathcal{F} \text{ with } i_S^!\mathcal{F} \in D_S^{\geq -\dim S} \\ \text{for all strata } S \end{array} \right\} $$

We also have the following stalkwise description of this t structure:

$$ {}^{p}D_X^{\leq 0} = \left\{ \begin{array}{l} \mathcal{F} \text{ such that} \\ \quad \dim_{\mathbb{R}} \operatorname{supp} H^i(\mathcal{F}) \leq -i \\ \text{for all } i. \end{array} \right\} $$

$$ {}^{p}D_X^{\geq 0} = \left\{ \begin{array}{l} \mathcal{F} \text{ such that} \\ \quad \dim_{\mathbb{R}} \operatorname{supp} H^i(\mathbb{D}\mathcal{F}) \leq -i \\ \text{for all } i. \end{array} \right\} $$

We can also describe ${}^{p}D_X^{\geq 0}$ as the vanishing of the <u>cosupport</u> of $\mathcal{F}$, those $x \in X$ with $i_x^! \mathcal{F} \neq 0$.

$$ {}^{p}D_X^{\geq 0} = \left\{ \begin{array}{l} \mathcal{F} \text{ such that} \\ \quad \dim_{\mathbb{R}} \operatorname{cosupp} H^i(\mathcal{F}) \leq -i \\ \text{for all } i. \end{array} \right\} $$

This last description is our stratification independent definition of the perverse t structure, which is equivalent to being in the stratified perverse t structure for some stratification S.

Example/exercise:

$\mathbb{P}^1$ with $\mathbb{C} \sqcup pt$

Show that:

$$IC(U, \underline{\mathbb{C}}) \cong \underline{\mathbb{C}}_X[1]$$

$$IC(pt, \underline{\mathbb{C}}) \cong \underline{\mathbb{C}}_{pt}$$

Show that $j_! \underline{\mathbb{C}}_U$, $j_* \underline{\mathbb{C}}_U$ are both perverse, and compute their composition series.

Similarly, compute the composition series of

$$\begin{matrix} V \\ \uparrow j \\ W \end{matrix} \quad \text{in deg } -1.$$

(If this isn't an inclusion, not perverse).