The intermediate extension functor.

Def. Table of stalks

$X = \bigsqcup_{s \in S} X_s$ is a strat.

Some local system since $\mathcal{F}$ is perverse

Strata	$d-1$	d	$d+1$
X_{s_n}		$H^d(\mathcal{F})\|_{X_{s_n}}$	
$X_{s_{n-1}}$			
$\vdots$			
X_{s_0}			

If $\mathcal{F} \subset P_X$ then $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ both have tables that look
perverse sheaves

$$\dim(X_{s_i}) \leq \dim(X_{s_{i+1}})$$

like

	$-\dim X_n = -\dim X$	$\cdots$		$-\dim X_2$	$-\dim X_1$	$-\dim X_0$
X_n	$*$	$\cdots$	0	0	0	0
$\vdots$	$*$ $\vdots$		$\vdots$ 0	$\vdots$	$\vdots$	$\vdots$
X_2	$*$	$\cdots$	$*$	0	$\vdots$	$\vdots$
X_1	$*$		$*$	$*$	0	0
X_0	$*$	$\cdots$	$*$	$*$	$*$	$*$

Recollements

For $i : Z \hookrightarrow X$ is closed embedding

$j : U = X \setminus Z \hookrightarrow X$ is open complement. Then

$$\begin{tikzcd}
D_Z \arrow[r, "i_* = i_!"] & D_X \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^* = j^!"] & D_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

"recollement + conditions"

Recall t-structure on D_X:

$$D_X^{\leq 0} = \{ \mathcal{F} \in D_X \mid i^* \mathcal{F} \in D_Z^{\leq 0},\ j^* \mathcal{F} \in D_U^{\leq 0} \}$$

$$D_X^{\geq 0} = \{ \mathcal{F} \in D_X \mid i^! \mathcal{F} \in D_Z^{\geq 0},\ j^! \mathcal{F} \in D_U^{\geq 0} \}$$

From this defn:

(i) i_* and j^* are t-exact.

(ii) i^* and $j_!$ are right t-exact (preserves $D^{\leq 0}$).

(iii) $i^!$ and j_* are left t-exact (preserves $D^{\geq 0}$).

So we have functors between perverse sheaves,

$$\begin{tikzcd}
P_Z \arrow[r, "i_* = i_!"] & P_X \arrow[l, bend right, "{}^p i^* = \tau^{\geq 0} i^*"'] \arrow[l, bend left, "{}^p i^! = \tau^{\leq 0} i^!"] \arrow[r, "j^* = j^!"] & P_U \arrow[l, bend right, "{}^p j_! = \tau^{\geq 0} j_!"'] \arrow[l, bend left, "{}^p j_* = \tau^{\leq 0} j_*"]
\end{tikzcd}$$

$$i^* : P_Z \xrightarrow{i^*} D_X^{\leq 0} \xrightarrow{\tau^{\geq 0}} D_X^{\leq 0} \cap D_X^{\geq 0}$$

Ex. (i) we have adjunctions, $({}^p i^*, i_* = i_! = {}^p i^!)$

& $({}^p j_!, j^* = j^!, {}^p j_*)$.

(ii) $i_*, {}^p j_!, {}^p j_*$ are fully faithful.

More facts

① $i_* : P_Z \longrightarrow P_X$ gives an isomorphism

$$i_* : P_Z \xrightarrow{\sim} \left(\begin{array}{c} \text{full subcat of } P_X \text{ of} \\ \text{object supported on } Z \\ \text{i.e. } \operatorname{Supp} \mathcal{F}^\bullet \subset Z \end{array} \right) = P^Z$$

where $\operatorname{supp} \mathcal{F}^\bullet = \{ x \in X \mid \underbrace{\mathcal{F}^\bullet_x}_{\text{stalk of complex}} \text{ is not acyclic} \}$

not acyclic → means 0 in D_X

② P^Z is a Serre subcategory of P_X i.e.

for any s.e.s. $0 \longrightarrow \mathcal{F}' \to \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ in P_X:

$\mathcal{F} \in P^Z$ iff $\mathcal{F}'$ and $\mathcal{F}''$ are in P^Z.

Moreover, $j^* : P_Z \longrightarrow P_U$ is the Serre quotient of P^Z.

i.e. $j^* : P_Z \longrightarrow P_U$ satisfies

(P) $j^*(\mathcal{F}) = 0$ for any $\mathcal{F} \in P^Z$

and any functor $F : P_X \longrightarrow \mathcal{E}$ satisfying (P)

factors through j^*:

$$\begin{tikzcd} P_X \arrow[r, "j^*"] \arrow[dr, "F"'] & P_U \arrow[d, dashed, "\exists"] \\ & \mathcal{E} \end{tikzcd}$$

Section 1.4 of BBD.

③ For $\mathcal{F} \in P_X$:

(i) $i_* {}^p i^* \mathcal{F}$ = largest quotient of $\mathcal{F}$ in P_X supported on Z

$i_! {}^p i^! \mathcal{F}$ = largest subobject of $\mathcal{F}$ in P_X supported on Z.

We saw P_Z embeds into P_X.

Does P_U embed into P_X?

<u>Guess:</u>

$$\left.\begin{array}{ll} j_* P_U & v \\ j_! P_U & v \end{array}\right\} \text{ they embed in } P_X$$

Define P^U = full subcategory of P_X with no subobjects or quotients supported on Z.

$$P_U = \{ \mathcal{F} \in P_X \mid {}^p i^* \mathcal{F} = 0 \text{ and } {}^p i^! \mathcal{F} = 0 \}$$

<u>Theorem.</u> $j^* : P^U \longrightarrow P_U$ is an equivalence of categories, with quasi-inverse $j_{!*} : P_U \longrightarrow P^U$ defined as follows:

(Step 1) For $\mathcal{F} \in P_U$:

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{F}, \overbrace{j^* {}^p j_*}^{\simeq \mathrm{id}} \mathcal{F}) \simeq \mathrm{Hom}({}^p j_! \mathcal{F}, {}^p j_* \mathcal{F})$$

$$1_{\mathcal{F}} \longmapsto \overline{1_{\mathcal{F}}}$$

(Step 2) $j_{!*} \mathcal{F} = \mathrm{im}\left(\overline{1_{\mathcal{F}}} : {}^p j_! \mathcal{F} \longrightarrow {}^p j_* \mathcal{F} \right)$

Intermediate or middle extension functor.

<u>Exercise</u>: Show $j^* : P^U \longrightarrow P^U$ is fully faithful.

Why is it essentially surjective?

For any $\mathcal{F} \in P_U$, $\quad j^* j_{!*} \mathcal{F} \simeq \mathcal{F}$

j^* exact (commutes with im)

<u>Why?</u>

$$j^* j_{!*} \mathcal{F} = j^* \operatorname{im}\left(j_! \mathcal{F} \longrightarrow j_* \mathcal{F}\right) = \operatorname{im}\left(\underbrace{j^* j_!}_{\simeq \, id} \mathcal{F} \longrightarrow \underbrace{j^* j_*}_{\simeq \, id} \mathcal{F}\right)$$

$$= \operatorname{im}\left(\mathcal{F} \xrightarrow{id} \mathcal{F}\right) = \mathcal{F}.$$

<u>Corollary</u>: Simple objects of P_X are either of the form:

(i) $i_* \mathcal{L}$ for $\mathcal{L} \in P_Z$ simple.

(ii) $j_{!*} \mathcal{L}$ for $\mathcal{L} \in P_U$ simple.

All simple objects on P_X are intermediate extensions of irred. local systems on the strata.

For a local system $\mathcal{L}$ on a strata X_s and

$h : X_s \hookrightarrow X$

locally closed

Write

$$IC(X, \mathcal{L}) := h_{!*} \mathcal{L}$$

Next week will compute examples