

The intermediate extension functor.

Def. Table of stalks

$X = \bigsqcup_{s \in S} X_s$ is a strat.

same local system since $\mathcal{F}$ is perverse

Strata	$d-1$	d	$d+1$
X_{S_n}		$H^d(\mathcal{F})$ X_{S_n}	
$X_{S_{n-1}}$			
$\vdots$			
X_{S_0}			

If $\mathcal{F} \in \mathcal{P}_X$ then $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ both have stalks that look like

$\uparrow$
perverse
degrees

$$\dim(X_{S_i}) \leq \dim(X_{S_{i+1}})$$

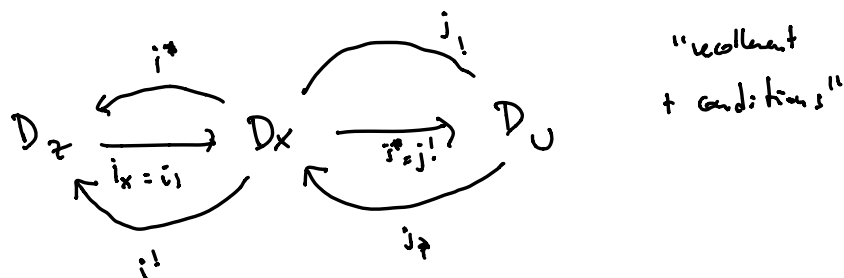
like

	$-\dim X_n = -\dim X$	$\dots$		$-\dim X_2$	$-\dim X_1$	$-\dim X_0$
X_n	*	$\dots$	0	0	0	0
$\vdots$	*		$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_2	*	$\dots$	0	0	$\vdots$	$\vdots$
X_1	*		*	*	0	0
X_0	*	$\dots$	*	*	*	*

Recollections

For $i: Z \hookrightarrow X$ is closed embedding

$j: U = X \setminus Z \hookrightarrow X$ is open complement. Then



Recall t-structure on D_X :

$$D_X^{\leq 0} = \{ f \in D_X \mid i^* f \in D_Z^{\leq 0}, j^* f \in D_U^{\leq 0} \}$$

$$D_X^{\geq 0} = \{ f \in D_X \mid i_! f \in D_Z^{\geq 0}, j_! f \in D_U^{\geq 0} \}$$

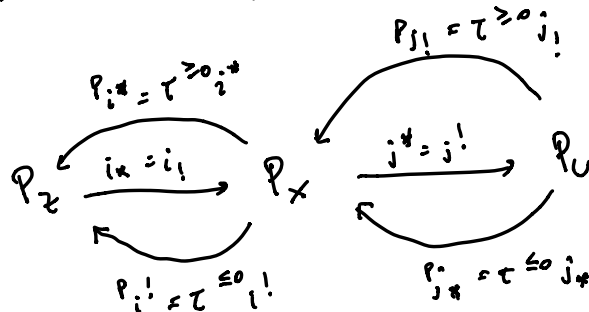
From this left:

(i) i_* and i^* are t-exact.

(ii) i^* and $j_!$ are right t-exact (preserve $D^{\leq 0}$).

(iii) $i_!$ and j^* are left t-exact (preserve $D^{\geq 0}$).

So we have functors between perverse sheaves,



$$i^*: P_Z \xrightarrow{i^*} D_X^{\leq 0} \xrightarrow{\tau^{\geq 0}} D_X^{< 0} \cap D_X^{\geq 0}$$

Ex. (i) we have adjunctions, $(i_!^*, i_* = i_! = i_!^!)$
 $\& (i_!^!, j^* = j^!, j_!^*)$.

(ii) $i_!^*, j_!^*, j_!^*$ are fully faithful.

More facts

① $i_* : P_Z \longrightarrow P_X$ gives an isomorphism

$$i_* : P_Z \xrightarrow{\sim} \left(\begin{array}{l} \text{full subcat of } P_X \text{ of} \\ \text{object supported on } Z \\ \text{i.e. } \text{Supp } \mathcal{F} \subset Z \end{array} \right) = P^Z$$

where $\text{supp } \mathcal{F} = \{x \in X \mid \mathcal{F}_x \text{ is not cyclic}\}$
stalk of complex → means 0 in D_X

② P^Z is a Serre subcategory of P_X i.e.

for any s.e.s. $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ in P_X :

$\mathcal{F} \in P^Z$ iff $\mathcal{F}'$ and $\mathcal{F}''$ are in P^Z .

Moreover, $j^* : P_Z \longrightarrow P_U$ is the Serre quotient of P^Z .

i.e. $j^* : P_Z \longrightarrow P_U$ satisfies

$$(P) \quad j^*(\mathcal{F}) = 0 \text{ for any } \mathcal{F} \in P^Z$$

and any functor $F : P_X \longrightarrow \mathcal{C}$ satisfying (P)

factors through j^* :

$$\begin{array}{ccc} P_X & \xrightarrow{j^*} & P_U \\ & \searrow F & \nearrow E \\ & \mathcal{C} & \end{array}$$

Section 1.4 of
BBD.

③ For $Z \in P_X$:

(i) $i_* p_i^* Z =$ largest quotient of Z in P_X supported on Z

$i_! p_i^! Z =$ largest subobject of Z in P_X supported on Z .

We saw P_Z embeds into P_X .

Does P_U embed into P_X ?

Guess: $j_* P_U \quad \textcircled{\vee}$
 $j_! P_U \quad \textcircled{\vee}$ } May embed in P_X

Define $P^U =$ full subcategory of P_X with no subobjects or quotients supported on Z .

$$P^U = \{ Z \in P_X \mid p_i^* Z = 0 \text{ and } p_i^! Z = 0 \}$$

Then.

$j^* : P^U \longrightarrow P_U$ is an equivalence of categories, with

quasi-inverse $j_!^* : P_U \longrightarrow P^U$ defined as follows:

(Step 1) For $Z \in P_U$:

$$\text{Hom}(Z, Z) \simeq \text{Hom}(Z, \overset{\text{id}}{j^* p_j^* Z}) \simeq \text{Hom}(p_j^! Z, i^* Z)$$

$$1_Z \longmapsto \overline{1_Z}$$

$$(\text{Step 2}) \quad j_!^* Z =: \text{im} \left(1_Z : p_j^! Z \longrightarrow p_j^* Z \right)$$

Intermediate or middle extension functor.

Exercise: Show $j^*: P^U \rightarrow P^U$ is fully faithful.

Why is it essentially surjective?

For any $\mathcal{F} \in P_U$, $j^* j_{!*} \mathcal{F} \simeq \mathcal{F}$

j^* exact (commutes with im)

Why?

$$\begin{aligned} j^* j_{!*} \mathcal{F} &= j^* \text{im} (j_! \mathcal{F} \rightarrow j_* \mathcal{F}) = \text{im} (\underbrace{j^* j_!}_{\text{id}} \mathcal{F} \rightarrow \underbrace{j^* j_*}_{\text{id}} \mathcal{F}) \\ &= \text{im} (\mathcal{F} \xrightarrow{\text{id}} \mathcal{F}) = \mathcal{F}. \end{aligned}$$

Corollary: Simple objects of P_X are either of the form:

(i) $i_* L$ for $L \in P_Z$ simple.

(ii) $j_{!*} L$ for $L \in P_U$ simple.

All simple objects on P_X are intermediate extensions of mod. local systems on the strata.

For a local system L on a stratum X_S and

$$h: X_S \hookrightarrow X$$

↑ locally closed

With

$$\text{IC}(X, L) := h_{!*} L$$

Next week will compute examples