Eg 5.2

I Formulate the problem.

Derivation of series for $\sin^2 x$.

1. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2$$

$$= x^2 - 2\frac{x^4}{3!} + \left\{\frac{x^6}{(3!)^2} + 2\frac{x^6}{5!}\right\} + \cdots$$

$$= x^2 - \frac{2}{3!}x^4 + \left\{\frac{1}{(3!)^2} + \frac{2}{5!}\right\}x^6 + \cdots$$

2. $$\sin 0 = 0$$

$$\frac{d}{dx}\sin^2 x = 2\sin x\cos x = \sin 2x$$

$$= 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \cdots$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{(2x)^{2k+1}}{(2k+1)!}$$

Integrate from $x=0$ to x :

$$\sin^2 x - \underset{\substack{\| \\ 0}}{\sin^2 0} = \int_0^x \frac{d}{dt}\sin^2 t \, dt$$

$$= \int_0^x \sum_{k=0}^{\infty} (-1)^k \frac{(2t)^{2k+1}}{(2k+1)!} \, dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1}}{(2k+1)!} \int_0^x t^{2k+1} \, dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1}}{(2k+1)!} \frac{x^{2k+2}}{2k+2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} 2^{2n-1} x^{2n} \quad (n = k+1)$$

$$= \sum_{n=1}^{\infty} u_n$$

$$u_n := \frac{(-1)^{n+1}}{(2n)!} 2^{2n-1} x^{2n}.$$

"is defined by"

We can derive a recurrence formula for u_n:

$$\frac{u_n}{u_{n-1}} = \frac{\dfrac{(-1)^{n+1}}{(2n)!} 2^{2n-1} x^{2n}}{\dfrac{(-1)^n}{(2n-2)!} 2^{2n-3} x^{2n-2}}$$

$$= -\frac{4x^2}{2n(2n-1)}$$

$$= -\frac{x^2}{n(n-\frac{1}{2})}$$

$$\left\{\boxed{\begin{array}{ll} u_n = -\dfrac{x^2}{n(n-\frac{1}{2})} u_{n-1} & , \quad n = 2, 3, 4, \ldots \\ u_1 = x^2 & \end{array}}\right.$$

From $n=1$, we need u_0: u_0 is not strictly defined but $n=1$ gives

$$x^2 = u_1 = -\frac{x^2}{1(1-\frac{1}{2})} u_0 = -2x^2 u_0$$

ie $u_0 = -\frac{1}{2}$.

$$\left\{\boxed{\begin{array}{ll} u_n = -\dfrac{x^2}{n(n-\frac{1}{2})} u_{n-1} & , \quad n = 1, 2, 3, \ldots \\ u_0 = -\frac{1}{2} & \end{array}}\right.$$

Define $S_n = \sum_{k=1}^{n} u_k$, ie the nth partial sum. The sequence for those is

$$\boxed{S_n = S_{n-1} + u_n} \quad n = 1, 2, 3, \ldots$$

$$S_0 = 0.$$

This is stage I.

II/ Coarse pseudo-code.

$u_1 = x^2, \quad S_1 = u_1$

Do $n = 2, N$

$$u_n = -\frac{x^2}{n(n-\frac{1}{2})} u_{n-1}$$

$$S_n = S_{n-1} + u_n$$

ENDDO

! OUTPUT RESULTS

Alternatives

1/ $u_0 = -\frac{1}{2}, \quad S_0 = 0$

Do $n = 1, N$

$$u_n = -\frac{x^2}{n(n-\frac{1}{2})} u_{n-1}$$

$$S_n = S_{n-1} + u_n$$

ENDDO

! OUTPUT RESULTS

2/

```
Do n=1,N
IF (n==1) THEN
```

$u_1 = x^2$

$S_1 = u_1$

```
ELSE
```

$$u_n = - \frac{x^2}{n(n-\frac{1}{2})} u_{n-1}$$

$S_n = S_{n-1} + u_n$

```
ENDIF
ENDDO
! OUTPUT RESULTS
```

III/ Skip

IV/ PROGRAM Sinxsq series

```
! Description: evaluate (sin x)^2 using
! its Taylor series about x=0.
! Declare variables:

INTEGER :: n, nmax = 7
REAL :: x, S, sinxsq, abserror

! Header:
WRITE(*,*) '            COMPARISON ...       '
WRITE(*,*) ' NUMBER       SERIES    ...      '
WRITE(*,*) ' OF TERMS            ...         '

! Prompt for x:
WRITE(*,*) 'Enter x:'
READ (*,*) x
! Exact value:

sinxsq = sin(x)**2
! Initialise loop:
```

```
u = x*x     ! or x**2
S = u
! Main loop :
Do n=2, nmax          ! N = nmax
u = - x*x*u/(n*(n-0.5))
S = S+u
abserror = abs (S - sinxsq)
WRITE (*, 10) n, S, sinxsq, abserror
10 FORMAT (I5, T20, F8.5, T30, F8.5, T40, E11.5)
ENDDO
WRITE (*,*) 'Series = ', S
STOP
END
```

9

```
      PROGRAM SINESQUARED
!     Partial sums of series for sine squared.

!     Initializations
!         X=2
          PRINT*,'ENTER X-VALUE:'
          READ*,X
          SIN2X=SIN(X)**2
          X2=X*X
          SUM=0

!     Headings
          PRINT 10
10        FORMAT('               COMPARISON OF VALUES OF SINE-SQUARED'//&
                 '  NUMBER                SERIES          INTRINSIC        ABSOLUTE'/&
                 ' OF TERMS              SUMMATION         FUNCTION       DIFFERENCE')
!     Cols        1234567890123456789012345678901234567890123456789012345678901234567890

!     Form partial sums and print requisite values
          DO 1 N=1,1000
              IF(N.EQ.1)THEN
                  TERM=X2
                  SUM=TERM
              ELSE
                  TERM=-TERM*2*X**2/(N*(2*N-1))
                  IF(ABS(TERM)<ABS(SUM)*1E-6)  STOP
                  SUM=SUM+TERM
              ENDIF
1         PRINT 100, N, SUM, SIN2X, ABS(SIN2X-SUM)
100       FORMAT(T4,I3,T16,F20.5,T31,F12.5,T46,F12.5)

      END

! ENTER X-VALUE:
!15
!              COMPARISON OF VALUES OF SINE-SQUARED
!
!
!  NUMBER                SERIES          INTRINSIC        ABSOLUTE
! OF TERMS              SUMMATION         FUNCTION       DIFFERENCE
!     1                     225.          0.42287       224.57713
!     2                  -16650.          0.42287     16650.42383
!     3                  489600.          0.42287    489599.56250
!     4                -7646560.          0.42287    ************
!     5                73715048.          0.42287    ************
!     6              -481023200.          0.42287    ************
!     7              2262187776.          0.42287    ************
!     8             -8024853504.          0.42287    ************
!     9             22231150592.          0.42287    ************
!    10            -49427812352.          0.42287    ************
!    11             90167574528.          0.42287    ************
!    12           -137433595904.          0.42287    ************
!    13            177706483712.          0.42287    ************
!    14           -197460279296.          0.42287    ************
!    15            190643290112.          0.42287    ************
!    16           -161466793984.          0.42287    ************
!    17            120974426112.          0.42287    ************
!    18            -80769302528.          0.42287    ************
!    19             48369639424.          0.42287    ************
!    20            -26133594112.          0.42287    ************
!    21             12805378048.          0.42287    ************
!    22             -5717389312.          0.42287    ************
!    23              2335987712.          0.42287    ************
!    24              -876795648.          0.42287    ************
!    25               303410432.          0.42287    ************
!    26               -97112000.          0.42287    ************
!    27                28838448.          0.42287    ************
!    28                -7965252.          0.42287    ************
!    29                 20539[illegible].          0.42287    ************
!    30                 -493339.          0.42287    493339.43750
!    31                  112826.          0.42287    112826.26563
!    32                  -22478.          0.42287     22478.58008
!    33                    5907.          0.42287      5907.05176
!    34                     300.          0.42287       299.70602
!    35                    1344.          0.42287      1344.55310
!    36                    1161.          0.42287      1160.60120
!    37                    1191.          0.42287      1191.24854
!    38                    1186.          0.42287      1186.40955
!    39                    1187.          0.42287      1187.13464
!    40                    1187.          0.42287      1187.03137
!    41                    1187.          0.42287      1187.04541
!    42                    1187.          0.42287      1187.04358
```

~10^11