4. **Taylor's Theorem for Functions of One Variable**: Suppose that $f \in C^n[a,b]$ and the derivative $f^{(n+1)}$ exists on $[a,b]$, and let $x_0 \in [a,b]$. For every $x \in [a,b]$ there exists $\eta(x)$ between x_0 and x such that

$$f(x) = f(x_0)+f^{(1)}(x_0)(x-x_0)+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\eta)}{(n+1)!}(x-x_0)^{n+1}\,.$$

Proof Use integration by parts:

$$\begin{aligned}
f(x) &= f(x_0) + \int_{x_0}^{x} f^{(1)}(t)\,dt \\
&= f(x_0) - \Big[f^{(1)}(t)(x-t)\Big]_{x_0}^{x} + \int_{x_0}^{x} f^{(2)}(t)(x-t)\,dt \\
&= f(x_0) - \left[f^{(1)}(t)(x-t) + f^{(2)}(t)\frac{1}{2}(x-t)^2 + \ldots + f^{(n)}(t)\frac{1}{n!}(x-t)^n\right]_{x_0}^{x} \\
&\quad + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt \\
&= f(x_0) + f^{(1)}(x_0)(x-x_0) + f^{(2)}(x_0)\frac{1}{2}(x-x_0)^2 + \ldots + f^{(n)}(x_0)\frac{1}{n!}(x-x_0)^n \\
&\quad + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt\,.
\end{aligned}$$

The remainder term is in Young's integral form. Noting that $(x-t)^n$ has one sign on $[x_0, x]$, application of the generalised integral mean value theorem gives

$$\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt = f^{(n+1)}(\eta)\frac{1}{n!}\int_{x_0}^{x}(x-t)^n\,dt = \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-x_0)^{n+1}\,.$$

5. **Taylor's Theorem for Functions of Two Variables**: Suppose that $f(x,y)$ and its partial derivatives of all orders less than or equal to $n+1$ are continuous on $D = \{(x,y) \mid a \leq x \leq b\,,\ c \leq y \leq d\}$ and let $(x_0, y_0) \in D$. For every $(x,y) \in D$, there exists ξ between x and x_0, and η between y and y_0 such that

$$\begin{aligned} f(x,y) &= f_0 + \left(\frac{\partial f}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0) + \frac{1}{2!}\left\{\left(\frac{\partial^2 f}{\partial x^2}\right)_0 (x-x_0)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 (x-x_0)(y-y_0)\right. \\ &\quad \left. + \left(\frac{\partial^2 f}{\partial y^2}\right)_0 (y-y_0)^2\right\} + \cdots + \frac{1}{n!}\left\{\sum_{j=0}^{n} \binom{n}{j} \left(\frac{\partial^n f}{\partial x^{n-j}\partial y^j}\right)_0 (x-x_0)^{n-j}(y-y_0)^j\right\} + R_n(x,y) \end{aligned}$$

where a subscript zero on f and its derivatives denotes evaluation at (x_0, y_0) and R_n is the remainder,

$$R_n(x,y) = \frac{1}{(n+1)!}\left\{\sum_{j=0}^{n+1} \binom{n+1}{j} \left(\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}\right)_{(\xi,\eta)} (x-x_0)^{n+1-j}(y-y_0)^j\right\}.$$

Proof Apply the one-dimensional Taylor's theorem about $t=0$ to

$$F(t) := f(x_0 + t(x-x_0), y_0 + t(y-y_0))\,,$$

noting that

$$\begin{aligned} F(0) &= f(x_0, y_0) \\ F^{(1)}(0) &= f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &\vdots \end{aligned}$$

6. **Taylor's Theorem for n Functions of n Variables**: Taylor's theorem for functions of two variables can easily be extended to real-valued functions of n variables $x_1, x_2, \ldots, x_n$. For n such functions f_1, f_2, ..., f_n, their n separate Taylor expansions can be combined using matrix notation into a single Taylor expansion. Define the column vectors $\mathbf{x}$ and $\mathbf{f}$, and the Jacobian matrix $\mathbf{J}$ of $\mathbf{f}$ by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \qquad \mathbf{J}(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Then the first two terms of the Taylor expansion of $\mathbf{f}$ about $\mathbf{x} = \mathbf{x}_0$ are

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \ldots .$$

7. **The O and o Symbols**. $f(x) = O(g(x))$ as $x \to x_0$ if there exists a constant C such that $|f(x)| \leq C|g(x)|$ for all x in some neighbourhood of x_0. $f(x) = o(g(x))$ as $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x) = 0$. E.g. $\sin x = O(x)$ and $\sin x = o(1)$ as $x \to 0$.

8. Peano Kernels (A).

Example 1.1 *The Trapezoidal Rule.*

The trapezoidal rule to approximate an integral:

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\{f(a)+f(b)\}\,.$$

The trapezoidal formula (trapezoidal rule with remainder term) for an integrand $f \in C^2[a,b]$:

$$\int_a^b f(x)\,dx = \frac{1}{2}(b-a)\{f(a)+f(b)\} - \frac{(b-a)^3}{12} f^{(2)}(\eta)\,.$$

The remainder $-(b-a)^3 f^{(2)}(\eta)/12$ quantifies the (truncation) error made in using the rule to approximate the integral of a C^2 integrand.

The trapezoidal formula for an integrand $f \in C^1[a,b]$:

$$\int_a^b f(x)\,dx = \frac{1}{2}(b-a)\{f(a)+f(b)\} + \int_a^b f^{(1)}(t)\left\{\frac{a+b}{2}-t\right\}\,dt\,.$$

The remainder

$$\int_a^b f^{(1)}(t)\left\{\frac{a+b}{2}-t\right\}\,dt$$

quantifies the error made in using the rule to approximate the integral of a C^1 integrand. The function $(a+b)/2-t$ in the remainder is known as a *Peano kernel.*