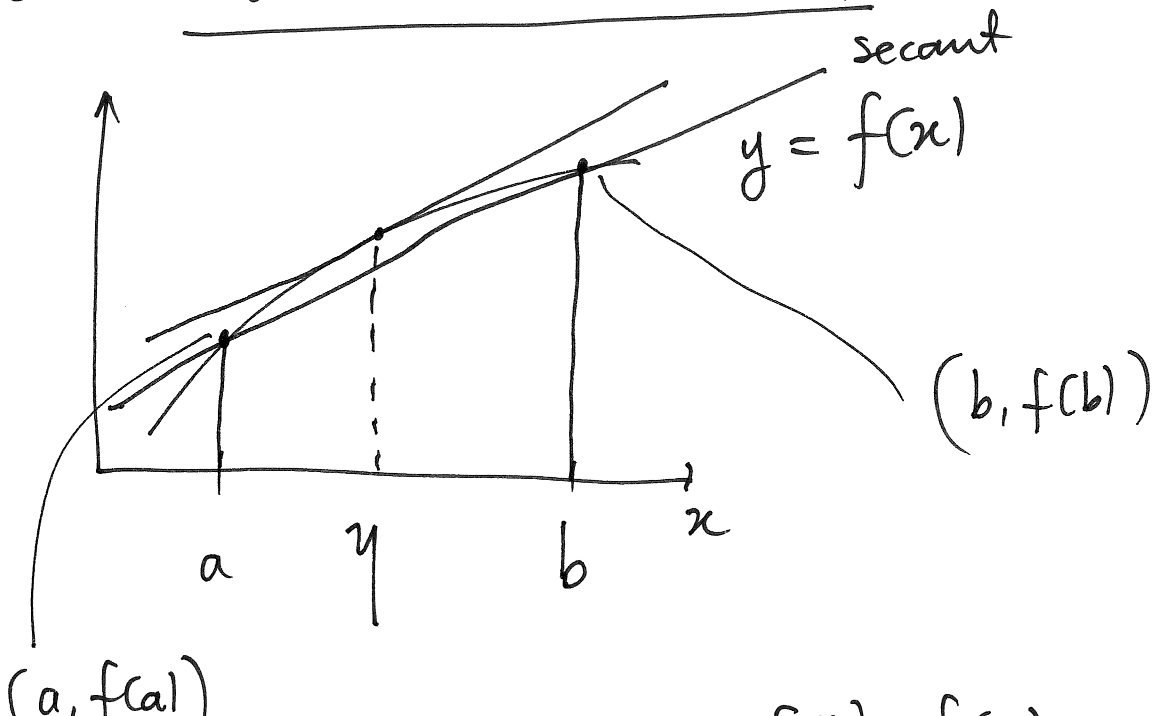
§1.2 Background Mathematics

2. Mean Value Theorem



$$\text{slope of secant} = \frac{f(b) - f(a)}{b - a}$$

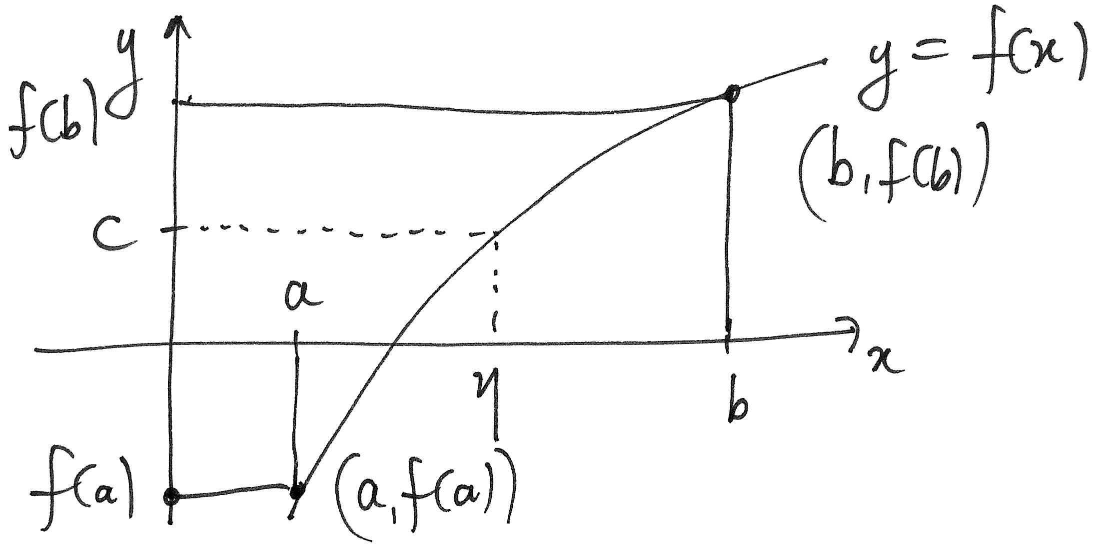
$$= \text{slope of tangent at } \eta, \quad a < \eta < b$$

$$= f'(\eta)$$

$$\boxed{f(b) = f(a) + f'(\eta)(b-a)}$$

$f \in C'[a,b]$, i.e. f is continuous on the interval $a \le x \le b$; $f' \equiv df/dx$ is continuous on $a \le x \le b$.

3. Intermediate Value Theorem



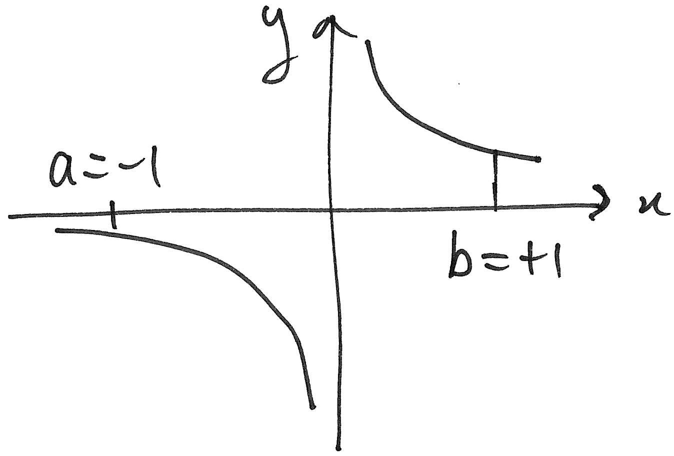
If $f \in C[a,b]$, i.e. <u>f is continuous</u> on $a \le x \le b$, and $f(a) < c < f(b)$, then there exists η, $a < \eta < b$, such that $f(\eta) = c$.

<u>Existence theorem</u> for zero of $f(x)$:

if $f(a)f(b) < 0$, i.e. f changes sign between a, b, then there exists x^*, $a < x^* < b$, such that $f(x^*) = 0$. This requires $f \in C[a,b]$.

__Counter-example__ : if f is not continuous this is false.

Eg $y = \frac{1}{x}$



$f(a) = -1$

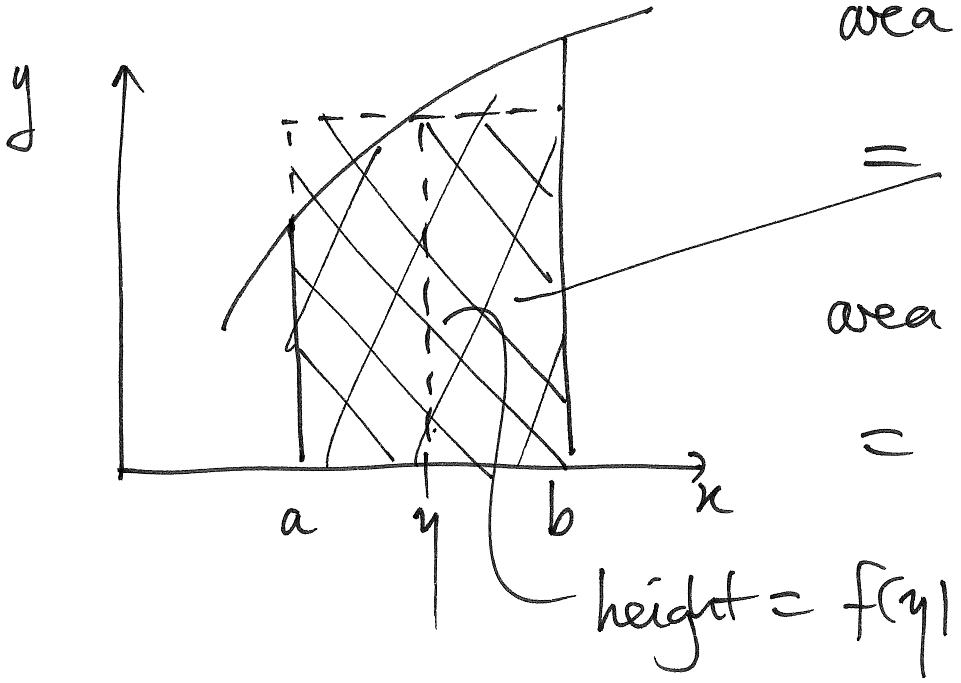
$f(b) = +1$

but $f(\eta) \neq 0$ for any η, $-1 < \eta < 1$.

Mean Value Theorem for Integral Calculus

Let $f \in C[a,b]$, then there exists η, $a < \eta < b$, such that

$$\int_a^b f(x)\,dx = f(\eta)(b-a)$$



area under curve

$$= \int_a^b f(x)\,dx$$

area of rectangle

$$= f(\eta)(b-a)$$

Generalised Theorem

Let $f \in C[a,b]$, $g \geq 0$ on $[a,b]$

There exists η, $a < \eta < b$, such that

$$\boxed{\int_a^b f(x)g(x)\,dx = f(\eta)\int_a^b g(x)\,dx}$$

Proof: The function f must satisfy

$$\min_{[a,b]} f \leq f(x) \leq \max_{[a,b]} f$$

for all $x \in [a,b]$. Multiply by $g(x) \geq 0$,

$$g(x)\min f \leq f(x)g(x) \leq g(x)\max f$$

Integrate

$$\min f \int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx$$
$$\leq \max f \int_a^b g(x)\,dx$$

Wlog $\int_a^b g(x)\,dx > 0$, so

$$\min_{[a,b]} f \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq \max_{[a,b]} f$$

By the Intermediate Value Theorem there exists η, $a < \eta < b$, such that

$$f(\eta) = \frac{\int_a^b f(x)\, g(x)\, dx}{\int_a^b g(x)\, dx}.$$

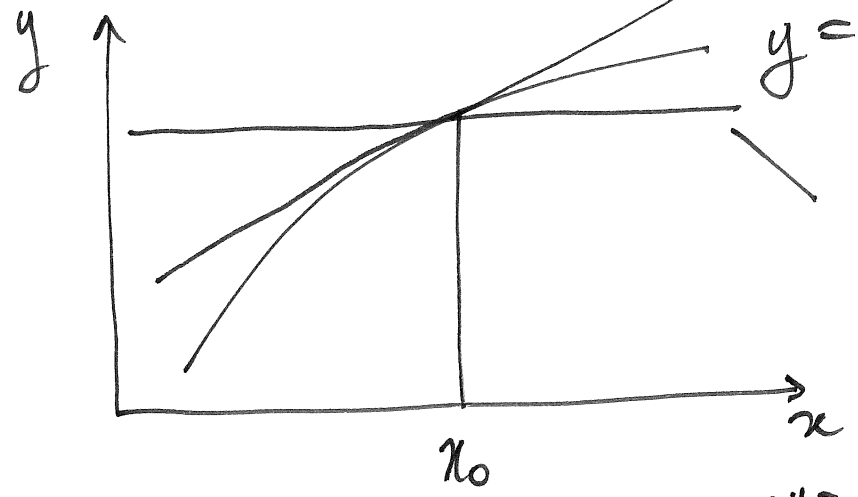
In these theorems the actual determination of η is usually extremely difficult.

Graphical Interpretation of Taylor Series

$y = f(x_0) + f'(x_0)(x - x_0)$

2 term Taylor expansion

$y = f(x)$

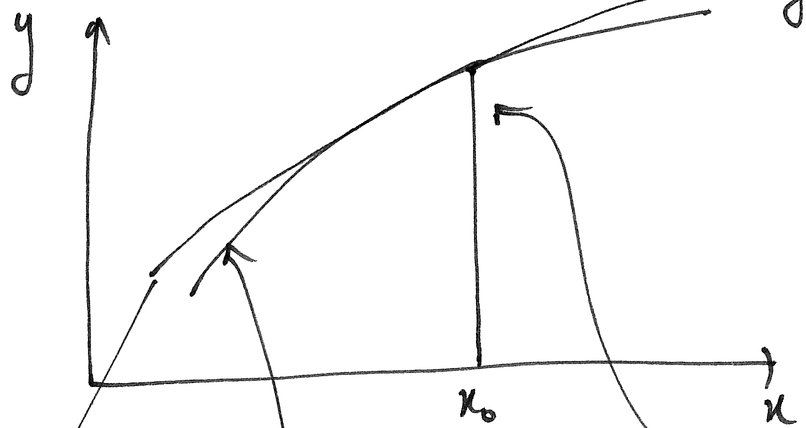


$y = f(x_0)$

1 term Taylor series

$y = f(x_0) + f'(x_0)(x - x_0)$
$+ \frac{1}{2} f''(x_0)(x - x_0)^2$

3 term Taylor expansion



parabola

poor approximation away from x_0

good approximation near x_0

Taylor polynomial are osculatory ("kiss")

Remarks

* Polynomials are easy to compute:

  $\times, +, -$ (no $\div$)

* Polynomials are easy to differentiate & integrate.

* Polynomial approximations may be easy to construct eg Taylor series.

* Other polynomial approximations are possible besides Taylor series: min-max approximations
  orthogonal polynomials

* More complicated functions than polynomials can be used to approximate other functions: rational functions, trigonometric functions (Fourier series), etc.

1D Example of Taylor Expansion

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f^{(2)}(x_0)(x-x_0)^2$$
$$+ \dots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$$
$$+ \frac{1}{(n+1)!} f^{(n+1)}(y)\,(x-x_0)^{n+1}$$

where y lies between x & x_0.

$f(x) = \sin x$, $x_0 = 1$, $n = 2$

$f(1) = \sin 1$

$f^{(1)}(x) = \cos x$ $\quad f^{(1)}(1) = \cos 1 \left(f' \equiv f^{(1)}\right)$

$f^{(2)}(x) = -\sin x$ $\quad f^{(2)}(1) = -\sin 1$

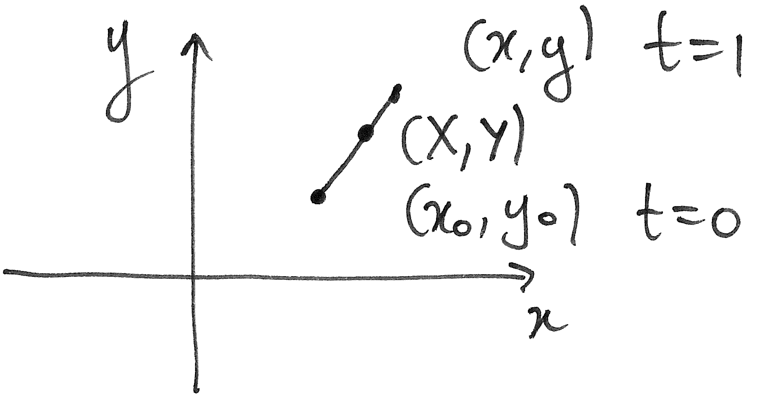
$f^{(3)}(x) = -\cos x$ $\quad f^{(3)}(y) = -\cos y$

Therefore the 3 term Taylor series for $\sin x$ about $x=1$ is

$$\sin x = \sin 1 + \cos 1\,(x-1) + \tfrac{1}{2}(-\sin 1)(x-1)^2$$
$$+ \tfrac{1}{6}(-\cos y)(x-1)^3$$

$$\boxed{\sin x = \sin 1 + \cos 1\,(x-1) - \tfrac{1}{2}\sin 1\,(x-1)^2 - \tfrac{1}{6}\cos y\,(x-1)^3}$$

Taylor Series for $F(x,y)$



Parametrise the line segment using t:

$$X = x_0 + t(x - x_0)$$
$$Y = y_0 + t(y - y_0)$$

$t=0$: $X = x_0$, $Y = y_0$

$t=1$: $X = x$, $Y = y$.

Define the 1D function

$$F(t) := f(X, Y)$$

"is defined by"

$$= f(x_0 + t(x - x_0), y_0 + t(y - y_0))$$

Apply the 1D Taylor theorem to $F(t)$:

$$F(t) = F(0) + \frac{dF}{dt}(0)t + \frac{1}{2}\frac{d^2F}{dt^2}(0)\,t^2 + \ldots$$

$$F(0) = f(x_0, y_0)$$

$$\frac{dF}{dt} = \frac{\partial f}{\partial X}\frac{dX}{dt} + \frac{\partial f}{\partial Y}\frac{dY}{dt}$$

$$= \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

$$\therefore \frac{dF}{dt}(0) = \frac{\partial f}{\partial x}(x_0, y_0)\,(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

etc.

Hence

$$F(t) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)\right] t + \ldots$$

$$F(1) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)\right] + \ldots$$

$$= f(x, y).$$

Taylor's theorem for 2 functions of 2 variables (to 2 terms)

$$f_1(x_1, x_2) = f_1(x_1^0, x_2^0) + \frac{\partial f_1}{\partial x_1}(x_1^0, x_2^0)(x_1 - x_1^0) + \frac{\partial f_1}{\partial x_2}(x_1^0, x_2^0)(x_2 - x_2^0) + \dots$$

by Taylor's theorem for 1 function of 2 variables.

Similarly,

$$f_2(x_1, x_2) = f_2(x_1^0, x_2^0) + \frac{\partial f_2}{\partial x_1}(x_1^0, x_2^0)(x_1 - x_1^0) + \frac{\partial f_2}{\partial x_2}(x_1^0, x_2^0)(x_2 - x_2^0) + \dots$$

$$\underline{f}(\underline{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} f_1(x_1^0, x_2^0) \\ f_2(x_1^0, x_2^0) \end{bmatrix} +$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial f_1}{\partial x_2}(x_1^0, x_2^0) \\ \frac{\partial f_2}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial f_2}{\partial x_2}(x_1^0, x_2^0) \end{bmatrix} \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{bmatrix} + \dots$$

$$= \underline{f}(\underline{x}_0) + \underline{\underline{J}}(\underline{x}_0)(\underline{x} - \underline{x}_0) + \dots$$

where $\underline{\underline{J}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$ is the Jacobian matrix of $\underline{f}$.