Damped Newton-Raphson method

We show there exists j such that

$$\| \underline{f}(\underline{x}^{(k)} + \underline{p}^{(k)}/2^{j-1}) \|_2 \geq \| \underline{f}(\underline{x}^{(k)}) \|_2$$

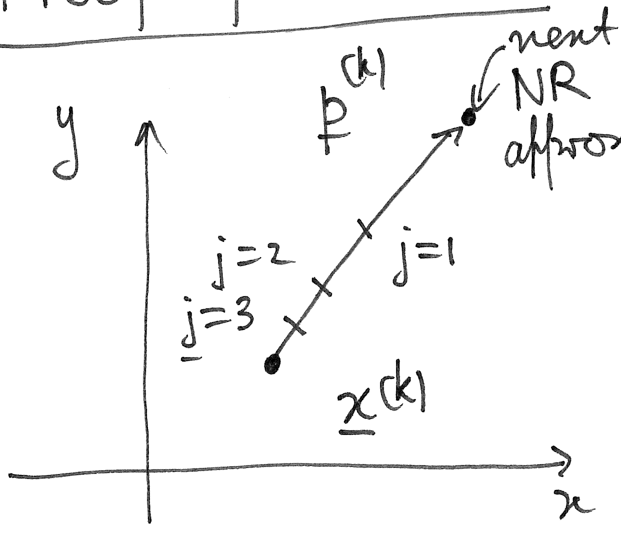
but

$$\| \underline{f}(\underline{x}^{(k)} + \underline{p}^{(k)}/2^{j}) \|_2 < \| \underline{f}(\underline{x}^{(k)}) \|_2$$

<u>where</u> $\quad \| \underline{f} \|_2^2 = f_1^2 + f_2^2 + \dots + f_n^2$

and $\quad \| \underline{f} \|_2 = \sqrt{f_1^2 + f_2^2 + \dots + f_n^2}$

Proof for $n=2$:



Suppose the current iterate is $\underline{x}^{(k)}$.
Let the NR ~~raph~~ correction be $\underline{p}^{(k)}$ where

$$\underline{J}^{(k)} \underline{p}^{(k)} = - \underline{f}^{(k)}$$

To prove a j exists we calculate the direction derivative of $\| \underline{f} \|_2^2$ at $\underline{x}^{(k)}$ in the direction $\underline{p}^{(k)}$.

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We show that there exists j such that

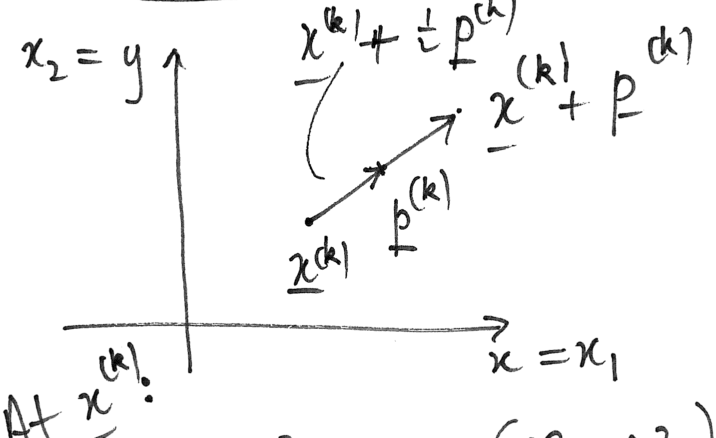
$$\| \underline{f}(\underline{x}^{(k)} + \underline{p}^{(k)}/2^{j-1}) \|_2 \geq \| \underline{f}(\underline{x}^{(k)}) \|_2 \quad \text{but}$$

$$\| \underline{f}(\underline{x}^{(k)} + \underline{p}^{(k)}/2^{j}) \|_2 < \| \underline{f}(\underline{x}^{(k)}) \|_2 .$$

$$\| \underline{f} \|_2^2 = f_1^2 + f_2^2 + \dots + f_n^2 \quad \text{or}$$

$$\| \underline{f} \|_2 = \sqrt{f_1^2 + f_2^2 + \dots + f_n^2} .$$

Proof for $n=2$: we show the directional derivative of $\| \underline{f} \|_2^2$ in the direction $\underline{p}^{(k)}$ of the NR method, is ~~vam~~ negative if $\| \underline{f} \|_2 \neq 0$



At $\underline{x}^{(k)}$:

$$\nabla \| \underline{f} \|_2^2 = \nabla (f_1^2 + f_2^2) \qquad (n=2)$$

$$= \left(\underline{i} \frac{\partial}{\partial x} + \underline{j} \frac{\partial}{\partial y} \right) (f_1^2 + f_2^2) \qquad x = x_1, \; y = x_2 .$$

$$= \underline{i} \frac{\partial f_1}{\partial x} 2 f_1 + \underline{i} \frac{\partial f_2}{\partial x} 2 f_2 + \underline{j} \frac{\partial f_1}{\partial y} 2 f_1 + \underline{j} \frac{\partial f_2}{\partial y} 2 f_2$$

$\Rightarrow$ the directional derivative in the direction of $\underline{p}^{(k)}$ is $\cancel{p}$ $\hat{\underline{p}}^{(k)} \cdot \nabla \| \underline{f} \|_2^2$

or $\cancel{\underline{p}^{(k)}}$ $\dfrac{\underline{p}^{(k)}}{|\underline{p}^{(k)}|} \cdot \nabla \| \underline{f} \|_2^2 = \dfrac{d}{ds} \| \underline{f} \|_2^2$

Here $\|\underline{f}\|_2^2 = f_1^2 + f_2^2 \quad (n=2)$.

Recall the directional derivative of a function g in the direction $\underline{d}$ is

$\hat{\underline{d}} \cdot \nabla g$. Here we need $\widehat{\underline{p}^{(k)}} \cdot \nabla \|f\|_2^2$

where $\widehat{\underline{p}^{(k)}} = \dfrac{\underline{p}^{(k)}}{|\underline{p}^{(k)}|}$. Since we are

only interested in the sign of the directional derivative, so consider

$$\underline{p}^{(k)} \cdot \nabla \|f\|_2^2 \Big|_{\underline{x} = \underline{x}^{(k)}}.$$

$$\nabla \|f\|_2^2 = \nabla (f_1^2 + f_2^2) = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y}\right)(f_1^2 + f_2^2)$$

$$= \underline{i}\left(2f_1 \frac{\partial f_1}{\partial x} + 2f_2 \frac{\partial f_2}{\partial x}\right) + \underline{j}\left(2f_1 \frac{\partial f_1}{\partial y} + 2f_2 \frac{\partial f_2}{\partial y}\right)$$

~~But~~ $\therefore$ $\underline{p}^{(k)} \cdot \nabla \|\underline{f}\|_2^2 = p_1^{(k)}\left(2f_1 \frac{\partial f_1}{\partial x} + 2f_2 \frac{\partial f_2}{\partial x}\right)$

$$+ p_2^{(k)}\left(2f_1 \frac{\partial f_1}{\partial y} + 2f_2 \frac{\partial f_2}{\partial y}\right)$$

since $\underline{p}^{(k)} = p_1^{(k)} \underline{i} + p_2^{(k)} \underline{j}$.