# Properties of Cond($\underline{\underline{A}}$)

① $\det(a\underline{\underline{A}}) = a^n \det \underline{\underline{A}}$ , $\underline{\underline{A}}$ $n \times n$

So in fact $\det \underline{\underline{A}}$ is not useful for examining the condition of $\underline{\underline{A}}\underline{x} = \underline{b}$. Recall if $\det \underline{\underline{A}} = 0$, then $\underline{\underline{A}}^{-1}$ does not exist & $\underline{\underline{A}}\underline{x} = \underline{b}$ may have no solutions.

$$\underline{\underline{A}}\underline{x} = \underline{b} \Rightarrow \varepsilon \underline{\underline{A}}\underline{x} = \varepsilon \underline{b} \qquad \varepsilon \neq 0$$

$$\det \varepsilon \underline{\underline{A}} = \varepsilon^n \det \underline{\underline{A}}$$

So we can make the coefficient $\underline{\underline{A}}$ have a determinant as small as we like by scaling the equations.

What about cond($\underline{\underline{A}}$)?

$$(\varepsilon \underline{\underline{A}})^{-1} = \varepsilon^{-1} \underline{\underline{A}}^{-1} \Rightarrow$$

$$\text{cond}(\varepsilon \underline{\underline{A}}) = \|\varepsilon \underline{\underline{A}}\| \, \|(\varepsilon \underline{\underline{A}})^{-1}\| \quad \text{(definition)}$$

$$= |\varepsilon| \, \|\underline{\underline{A}}\| \, \|\varepsilon^{-1} \underline{\underline{A}}^{-1}\|$$

$$= \cancel{|\varepsilon|} \, \|\underline{\underline{A}}\| \, \cancel{|\varepsilon^{-1}|} \, \|\underline{\underline{A}}^{-1}\|$$

$$= \text{cond}(\underline{\underline{A}})$$

So the condition number is invariant under scaling of $\underline{\underline{A}}$.

② $\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}$ (= identity matrix)

$$\|\underline{\underline{I}}\| = \|\underline{\underline{A}}\,\underline{\underline{A}}^{-1}\|$$

$$\leq \|\underline{\underline{A}}\| \, \|\underline{\underline{A}}^{-1}\|$$

$$= \text{cond}(\underline{\underline{A}})$$

$$\boxed{\text{cond}(\underline{\underline{A}}) \geq \|\underline{\underline{I}}\|}$$

$\|\underline{\underline{I}}\|_1 = 1 \qquad \|\underline{\underline{I}}\|_\infty = 1.$

③ Recall $\quad \|\underline{\underline{A}}\| = \max_{\underline{x} \neq \underline{0}} \dfrac{\|\underline{\underline{A}}\,\underline{x}\|}{\|\underline{x}\|}$

If $(\underline{x}_i, \lambda_i)$ is an eigenpair of $\underline{\underline{A}}$,

i.e. $\quad \underline{\underline{A}}\,\underline{x}_i = \lambda_i \underline{x}_i, \quad \underline{x}_i \neq \underline{0}.$

$$\|A\| \geq \frac{\|\underline{\underline{A}}\,\underline{x}_i\|}{\|\underline{x}_i\|} = \frac{\|\lambda_i \underline{x}_i\|}{\|\underline{x}_i\|}$$

$$= \frac{|\lambda_i|\,\cancel{\|\underline{x}_i\|}}{\cancel{\|\underline{x}_i\|}} = |\lambda_i|$$

So $\|A\| \geq |\lambda_i|$ for all eigenvalues of $\underline{\underline{A}}$,

or $\quad \boxed{\|A\| \geq \max_i |\lambda_i|} \quad (*)$

What about $\|\underline{\underline{A}}^{-1}\|$? If $(\underline{x}_i, \lambda_i)$ is an eigenpair of $\underline{\underline{A}}$, then $(\underline{x}_i, \lambda_i^{-1})$ is an eigenpair of $\underline{\underline{A}}^{-1}$, i.e. $\underline{\underline{A}}^{-1}\underline{x}_i = \lambda_i^{-1}\underline{x}_i$.

$\lambda_i \neq 0$ if $\underline{\underline{A}}^{-1}$ exists.

Apply $(*)$ to $\underline{\underline{A}}^{-1}$: $\quad \|\underline{\underline{A}}^{-1}\| \geq \dfrac{1}{\min_j |\lambda_j|}$

$$= \max_j \frac{1}{|\lambda_j|}$$

So

$$\text{cond}(\underline{\underline{A}}) = \|\underline{\underline{A}}\|\,\|\underline{\underline{A}}^{-1}\| \geq \frac{\max_i |\lambda_i|}{\min_j |\lambda_j|}$$

(4) Suppose $\underline{\underline{B}}$ is singular, i.e. $\underline{\underline{B}}^{-1}$ does not exist, so $\text{cond}(\underline{\underline{B}})$ does not exist.

Let $\underline{\underline{A}} = \underline{\underline{B}} + \varepsilon \underline{\underline{I}}$. Let the eigenvalues of $\underline{\underline{B}}$ be $\mu_i$ ~~as in (3)~~.

Then the eigenvalues of $\underline{\underline{A}}$ are

$\lambda_i = \mu_i + \varepsilon$. Since $\underline{\underline{B}}$ is singular $\min_i |\mu_i| = 0$. So

$$\min_i |\lambda_i| = |\varepsilon|$$

$$\max_i |\lambda_i| = \max_i |\mu_i + \varepsilon|$$

Hence

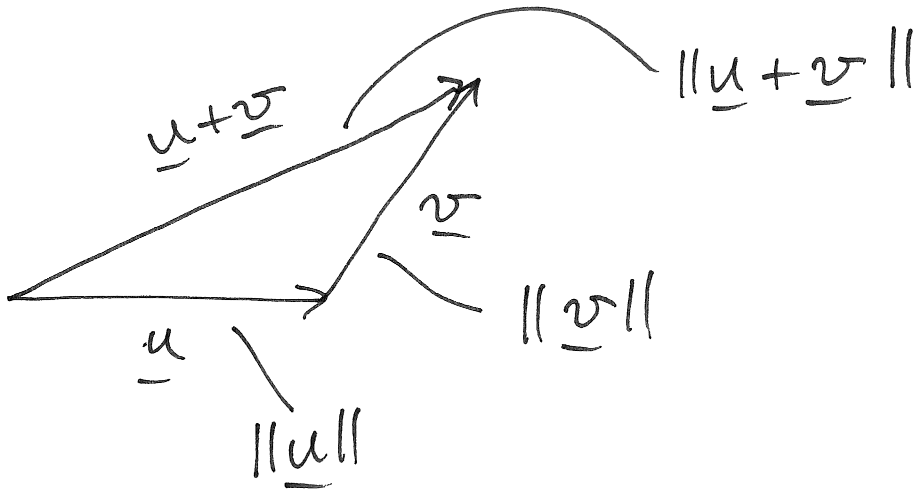
$$\text{cond}(\underline{\underline{A}}) = \text{cond}(\underline{\underline{B}} + \varepsilon \underline{\underline{I}})$$

$$\geq \frac{\max_i |\lambda_i|}{\min_j |\lambda_j|}$$

$$= \frac{\max_i |\mu_i + \varepsilon|}{|\varepsilon|}$$

$$\to \infty \quad \text{as} \quad \varepsilon \to 0$$

# Some properties of vector norms




$$\|\underline{u}+\underline{v}\| \leqslant \|\underline{u}\| + \|\underline{v}\|$$

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$$\|\underline{v}\|_p = \sqrt[p]{|\underline{v}_1|^p + |v_2|^p + \ldots + |v_n|^p}$$

$$\lim_{p\to\infty} \|\underline{v}\|_p = \|\underline{v}\|_\infty$$

For our purposes $\|\underline{v}\|_1$, $\|\underline{v}\|_\infty$ are the most important.