Computing Humbert Surfaces

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Outline

The Siegel upper half plane

Definition
The Siegel upper half plane of degree g is

$$\mathcal{H}_g = \{\tau \in \mathrm{Mat}_{g\times g}(\mathbb{C}) \mid {}^t\tau = \tau \ , \ \ \mathrm{Im}\,(\tau) > 0\}\,.$$

- Each $\tau \in \mathcal{H}_g$ corresponds to a PPAV $A_\tau/\mathbb{C}$ with period matrix $(\tau \ I_g) \in \mathrm{Mat}_{g\times 2g}(\mathbb{C})$.
- $A_\tau \cong A_{\tau'} \Leftrightarrow \exists M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\tau' = M \cdot \tau := (a\tau + b)(c\tau + d)^{-1}$.
- $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ is a moduli space for dimension g PPAV's.
- $\dim \mathcal{A}_g = \frac{1}{2}g(g+1)$. In particular, $\dim \mathcal{A}_2 = 3$ and $\mathcal{A}_2$ is called the Siegel modular threefold.

Extra endomorphisms

Let A be a PPAS ($g = 2$). Then $\mathrm{End}(A)$ is an order in $\mathrm{End}(A) \otimes \mathbb{Q}$ which isomorphic to one of the following algebras:

(0) quartic CM field
(1) indefinite quaternion algebra over $\mathbb{Q}$
(2) real quadratic field
(3) $\mathbb{Q}$

The irreducible components of the corresponding moduli spaces in $\mathcal{A}_2$ which have "extra endomorphisms" are known as

(0) CM points
(1) Shimura curves
(2) Humbert surfaces

Humbert's equation

Humbert showed that any $\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathcal{A}_2$ satisfying the equation

$$k\tau_1 + \ell\tau_2 - \tau_3 = 0$$

defines a Humbert surface H_Δ of discriminant $\Delta = 4k + \ell > 0$.

Example

$H_1 = \mathrm{Sp}_4(\mathbb{Z})\backslash \left\{ \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_3 \end{pmatrix} \right\} = \mathrm{Sp}_4(\mathbb{Z})\backslash \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \right\}$, the set of abelian varieties which split as a product of elliptic curves.

Task: Find "useful" algebraic models for H_Δ.

Algebraic models

- Torelli says that the map $C \mapsto \mathrm{Jac}(C)$ defines a birational map between $\mathcal{M}_2$, the moduli space of genus 2 curves and $\mathcal{A}_2$ (In fact $\mathcal{M}_2 \cong \mathcal{A}_2 - H_1$).
- The function field of $\mathcal{A}_2$ (and hence $\mathcal{M}_2$) is $\mathbb{C}(j_1, j_2, j_3)$ where j_i are the absolute Igusa invariants.
- There exists an irreducible polynomial $H_\Delta(j_1, j_2, j_3)$ whose zero set is the Humbert surface of discriminant Δ.

Unfortunately, working with j_i is impractical (enormous degrees, giant coefficients).
Solution: add some level structure.

Algebraic models

Runge's model

Runge uses level $\Gamma^*(2,4)$ structure, with four theta functions:

$$\theta\begin{bmatrix} a \\ (0,0) \end{bmatrix}(2\tau),\ a \in \mathbb{Z}^2/2\mathbb{Z}^2$$

where

$$\theta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau) = \sum_{x\in\mathbb{Z}^2} e^{2\pi i\left(\frac{1}{2}(x+\frac{m'}{2})\cdot\tau\cdot{}^t(x+\frac{m'}{2})+(x+\frac{m'}{2})\cdot{}^t(\frac{m''}{2})\right)}$$

are classical theta functions of half integral characteristics determined by values $m', m'' \in \mathbb{Z}^2/2\mathbb{Z}^2$.

Algebraic models

Rosenhain model

We use level $\Gamma(2)$-structure with three functions

$$
\begin{aligned}
\lambda_1(\tau) &= \left(\frac{\theta\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]}\right)^2 ,\\
\lambda_2(\tau) &= \left(\frac{\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]}\right)^2 ,\\
\lambda_3(\tau) &= \left(\frac{\theta\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]}\right)^2
\end{aligned}
$$

called Rosenhain invariants. These generate the function field of $\mathcal{A}_2(2) = \Gamma(2)\backslash\mathcal{H}_2$.

Relation to genus 2 curves

- Given a genus 2 curve $\mathcal{C}: \; y^2 = \prod_{i=1}^{6}(x - u_i)$ we can send three of the u_i to $0, 1, \infty$ via a fractional linear transformation to get an isomorphic curve with a Rosenhain model

$$y^2 = x(x-1)(x-t_1)(x-t_2)(x-t_3) \; .$$

  The t_i are called Rosenhain invariants.
- $(0, 1, \infty, t_1, t_2, t_3)$ determines an ordering of the Weierstrass points and a level 2 structure on $\mathrm{Jac}(\mathcal{C})$ ($\in \mathcal{A}_2(2)$).
- Let $\mathcal{M}_2(2)$ denote the moduli space of genus 2 curves together with a full level 2 structure. Points of $\mathcal{M}_2(2)$ are given by triples (t_1, t_2, t_3) where $t_i \neq t_j, 0, 1$ for all i, j.
- The forgetful morphism $\mathcal{M}_2(2) \to \mathcal{M}_2$ is a Galois covering of degree $720 = |S_6|$ where S_6 acts on the Weierstrass 6-tuple by permutations, followed by renormalising the first three to $(0, 1, \infty)$.

Runge's method

Let $\phi : \mathcal{A}' \to \mathcal{A}_2$ be a finite cover of $\mathcal{A}_2$. Then

$$\phi^{-1} H_\Delta = \bigcup_{\text{finite}} H_\Delta^{(i)} \,.$$

Given functions $\{f_i(\tau)\}_{i=1,\ldots,n}$ generating the function field of $\mathcal{A}'$, compute $H_\Delta^{(i)}(f_1, \ldots, f_n)$ as follows:

1. Calculate the degree of the Humbert components $H_\Delta^{(i)}$ (given by a formula).
2. Compute power series representations of the $f_i(\tau)$ restricted to $H_\Delta \subset \mathcal{H}_2$.
3. Solve $H_\Delta^{(i)}(f_1, \ldots, f_n) = 0$ in the power series ring (truncated series with large precision) using linear algebra.

We shall consider the level 2 covering $\mathcal{A}_2(2) \to \mathcal{A}_2$.

Step 1 - degree formula

Fortunately much arithmetic-geometric information is known about Humbert surfaces (van der Geer '82). The number of Humbert components in $\mathcal{A}_2(2)$ is

$$m(\Delta) = \begin{cases} 10 & \text{if } \Delta \equiv 1 \bmod 8 \\ 15 & \text{if } \Delta \equiv 0 \bmod 4 \\ 6 & \text{if } \Delta \equiv 5 \bmod 8 \end{cases}$$

(see Besser '98).

The degree of any Humbert component $H_\Delta^{(i)}$ in $\mathcal{A}_2(2)$ is given by a recursive formula

$$a_\Delta = \sum_{x>0} v(\Delta/x^2)m(\Delta/x^2)\deg\left(H^{(i)}_{\Delta/x^2}\right)$$

where

$$v(x) = \begin{cases} 1/2 & \text{if } x = 1 \\ 1 & \text{if } x \geq 2, x \equiv 0, 1 \bmod 4. \\ 0 & \text{otherwise} \end{cases}$$

Moreover, a_Δ is the coefficient of a certain modular form of weight $5/2$ for the group $\Gamma_0(4)$, which fortunately has a more elementary description due to a formula of Siegel:

$$a_\Delta - 24\sum_{x\in\mathbb{Z}} \sigma_1\left(\frac{\Delta - x^2}{4}\right) = \begin{cases} 12\Delta - 2 & \text{if } \Delta = \Box \\ 0 & \text{otherwise} \end{cases}$$

Here are the degrees for small discriminants:

Δ	1	4	5	8	9	12	13	16	17	20	21	24
$\deg(H_\Delta^{(i)})$	2	4	8	8	24	16	40	32	48	32	80	48
actual deg	1	2	8	8	16	16	40	24	48	32	80	48

Remarks

- When $\Delta \equiv 0 \pmod 4$ we have

$$\begin{aligned} m_{\mathrm{Runge}}(\Delta) &= 4 \times m_{\mathrm{Rosenhain}}(\Delta) \\ \Rightarrow \deg_{\mathrm{Runge}}(\Delta) &= \frac{1}{4} \times \deg_{\mathrm{Rosenhain}}(\Delta)\,. \end{aligned}$$

- In reality (after computing these equations) the actual degrees of $H_{n^2}^{(i)}(\lambda_j)$ are less than what the formula produces. For example

$$H_1 : \{e_i - e_j = 0, i \neq j\} \cup \{e_i = 0\} \cup \{e_i - 1 = 0\}\ .$$

Step 2 - power series

Write $\Delta = 4k + \ell$ where ℓ is either 0 or 1, and k is uniquely determined. The Humbert surface of discriminant Δ can be defined by the set

$$H_\Delta = \mathrm{Sp}_4(\mathbb{Z}) \backslash \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & k\tau_1 + \ell\tau_2 \end{pmatrix} \in \mathcal{H}_2 \right\}.$$

Restrict $\theta\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to H_Δ to get a Laurent series

$$\theta\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right](\tau) = \sum_{(x_1,x_2)\in\mathbb{Z}^2} e^{\pi i(x_1 c + x_2 d)} r^{(2x_1+a)^2 + k(2x_2+b)^2} q^{2(2x_1+a)(2x_2+b) + \ell(2x_2+b)^2}$$

where $r = e^{2\pi i \tau_1/8}$ and $q = e^{2\pi i \tau_2/8}$.

Unfortunately q has negative exponents. Substitute $r = pq$ to get

$$\sum_{(x_1,x_2)\in\mathbb{Z}^2} (-1)^{x_1c+x_2d} p^{(2x_1+a)^2+k(2x_2+b)^2} q^{(2x_1+a+2x_2+b)^2+(k+\ell-1)(2x_2+b)^2}$$

which is a power series with integer coefficients.
Using this representation one can compute the restriction of $\lambda_1, \lambda_2, \lambda_3$ to a Humbert surface as elements of $\mathbb{Z}[[p,q]]/(p^N, q^N)$ fairly easily.

Step 3 - linear algebra

Let $d = \deg(H_\Delta^{(i)})$. To find the algebraic relation $H_\Delta^{(i)}$:

- Compute all monomials of degree $\leq d$ in the variables e_1, e_2, e_3.
- Substitute $e_i = \lambda_i(p, q) \in \mathbb{Z}[[p, q]]/(p^N, q^N)$ in each monomial.
- Use linear algebra to find linear dependencies between the power series monomials $p^m q^n$ (compute null space of a big matrix).

- With high enough precision there will be exactly one linear relation between the monomials e_i. This produces the polynomial relation $H_{\Delta}^{(i)}(e_1, e_2, e_3) = 0$ which defines a Humbert component.
- Once one component has been determined, the others can easily be found by looking at the Rosenhain (S_6) orbit of a component.

These other components will turn out to be useful when we look at Shimura curves.

Runtime analysis

- There are:
  - $\binom{d+3}{3} = O(d^3)$ monomials to be evaluated
  - $O(N^2)$ coefficients of evaluated power series expressions of precision N.
- Runtime cost is dominated by the nullspace calculation: $O(d^6N^2) \geq O(d^9)$ to find a unique solution.
- Symmetries of the equation (arising from the fixed group of the humbert component) can be exploited to reduce the matrix size by a constant factor, giving a speedup by a constant factor.
- Not overly efficient, but least it's only a one time calculation..

Example

We calculate a component of H_5:

$$
\begin{aligned}
\lambda_1 &= 1 + 16p^4q^8 + O(p^{12}q^{12}) \\
\lambda_2 &= 1 + 4q^4 + 8q^8 - 8p^4q^4 - 24p^4q^8 + 4p^8q^8 + 48p^8q^8 + O(p^{12}q^{12}) \\
\lambda_3 &= 1 + 4q^4 + 8q^8 + 8p^4q^4 + 40p^4q^8 + 4p^8q^8 + 48p^8q^8 + O(p^{12}q^{12})
\end{aligned}
$$

Using power series with precision 65, we compute the Humbert component

$$
\begin{aligned}
&e_2^2e_3^2 - 2e_2^2e_3^3 + e_2^2e_3^4 + 2e_1e_2e_3^3 - 2e_1e_2e_3^4 - 2e_1e_2^2e_3 - 2e_1e_2^2e_3^2 + 4e_1e_2^2e_3^3 + 2e_1e_2^3e_3 \\
&-2e_1e_2^3e_3^3 + e_1^2e_3^4 - 2e_1^2e_2e_3^3 + e_1^2e_2^2 + 4e_1^2e_2^2e_3 - 4e_1^2e_2^2e_3^2 - 2e_1^2e_2^3 - 2e_1^2e_2^3e_3 \\
&+4e_1^2e_2^3e_3^2 + e_1^2e_2^4 - 2e_1^2e_2^4e_3 + e_1^2e_2^4e_3^2 - 2e_1^3e_3^3 - 2e_1^3e_2e_3 + 4e_1^3e_2e_3^2 + 2e_1^3e_2e_3^3 \\
&-2e_1^3e_2^2e_3^2 + 2e_1^3e_2^3e_3 - 2e_1^3e_2^3e_3^2 + e_1^4e_3^2 - 2e_1^4e_2e_3^2 + e_1^4e_2^2e_3^2
\end{aligned}
$$

Application: Computing Shimura Curves

Quaternion orders

bare essentials

Let R be an order in an indefinite $\mathbb{Q}$-quaternion algebra A.

- R is a QM-order if $R = \mathrm{End}(X)$ for some abelian surface X.
- Any $x \in A$ satisfies $x^2 - tx + n = 0$ where t, n are the reduced trace, norm respectively.
- $\Delta(x) = t(x)^2 - 4n(x)$ defines a discriminant form

$$\Delta(x, y) = \frac{1}{2}(\Delta(x + y) - \Delta(x) - \Delta(y)) \ .$$

- The discriminant $d(x_1, \ldots, x_4)$ of a module generated by $x_1, \ldots, x_4$ is defined to be the positive square root of

$$d(x_1, \ldots, x_4)^2 = -\det(t(x_i x_j)) \ .$$

Theorem (Runge '99)

1. *Any QM-order can be written as* $R = \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\alpha\beta$ *such that the discriminant matrix*

$$S_\Delta = \begin{pmatrix} \Delta(\alpha) & \Delta(\alpha,\beta) \\ \Delta(\alpha,\beta) & \Delta(\beta) \end{pmatrix}$$

   is positive definite. The discriminant of R *equals* $\det(S_\Delta)/4$*.*

2. *A change of basis corresponds to changing the discriminant matrix to* ${}^t g S_\Delta g$ *for some* $g \in \mathrm{GL}_2(\mathbb{Z})$*.* $\Rightarrow$ *can assume discriminant matrix is reduced.*

3. *If two orders have the same discriminant matrix which is primitive (gcd(entries)=1) then the corresponding Shimura curves are isomorphic.*

Theorem (Hashimoto '95, Runge '99)

Let $\mathcal{O} = \mathbb{Z}[\omega]$ be a quadratic order of discriminant Δ. Let S_Δ be a discriminant matrix of a QM order R. The following are equivalent:

1. *Δ is primitively represented by S_Δ.*
2. *There exists an embedding $\mathcal{O} \hookrightarrow R$ such that $R \cap \mathbb{Q}(\omega) = \mathcal{O}$.*
3. *A Shimura curve $\mathcal{C}$ with QM order R is contained in H_Δ.*

If we work in a finite cover, we have

$$\mathcal{C}^{(h)} \subset H^{(i)}_{\Delta(\alpha)} \cap H^{(j)}_{\Delta(\beta)}$$

if and only if we can write

$${}^t g S_\Delta g = \begin{pmatrix} \Delta(\alpha) & * \\ * & \Delta(\beta) \end{pmatrix}$$

for some $g \in \mathrm{GL}_2(\mathbb{Z})$.

Example

$H_5 \cap H_8$ contains four Shimura curves $\mathcal{C}_S$:

Discriminant matrix S	QM-order discriminant $= \det(S)/4$
$\begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix}$	10
$\begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$	9
$\begin{pmatrix} 5 & 4 \\ 4 & 8 \end{pmatrix} \sim \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$	6
$\begin{pmatrix} 5 & 6 \\ 6 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$	4

- This intersection was first computed by Hashimoto and Murabayashi (1995).

Computing intersections

- For intersections of Humbert components $H_{\Delta_i}(e_1, e_2, e_3)$ we can find plane affine models simply by taking resultants with respect to e_1.
- As we are working with coordinates in $\mathcal{M}_2(2) = \mathcal{A}_2(2) - H_1$, we will not be able to compute any Shimura curves in H_1.
- S_6 acts on Humbert components, hence acts on their intersections producing isomorphic curves.
- Take one curve from each S_6-orbit. Each of these intersections is a component of a Shimura curve $\mathcal{C}_S$ for some discriminant matrix S.

Example

In our $H_5 \cap H_8$ example, there are three non-equivalent intersections:

$$\begin{aligned} \mathcal{C}_1 &: \text{ a genus 1 curve} \\ \mathcal{C}_2 &: \text{ a genus 3 hyperelliptic curve} \\ \mathcal{C}_3 &: \text{ a genus 3 non-hyperelliptic curve} \end{aligned}$$

and the Shimura curves in $\mathcal{M}_2(2)$ are

$$\mathcal{C}_{\begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix}}, \mathcal{C}_{\begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}} \text{ and } \mathcal{C}_{\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}}$$

so there is a one-one correspondence between the $\mathcal{C}_i$ and the $\mathcal{C}_S$, to be determined.

Look at other Humbert intersections with "related" discriminants. Write $\mathcal{D}(a,b)$ for the set of discriminant matrices of QM-orders of Shimura curves in $H_a \cap H_b$. We have

$$\begin{aligned}
\mathcal{D}(5,5) &= \{ \left(\begin{smallmatrix} 5 & 1 \\ 1 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 2 \\ 2 & 5 \end{smallmatrix}\right) \} \\
\mathcal{D}(4,5) &= \{ \left(\begin{smallmatrix} 4 & 0 \\ 0 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 2 \\ 2 & 5 \end{smallmatrix}\right) \} \cup \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right) \} \\
\mathcal{D}(5,9) &= \{ \left(\begin{smallmatrix} 5 & 1 \\ 1 & 9 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 2 \\ 2 & 8 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 0 \\ 0 & 5 \end{smallmatrix}\right) \} \cup \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right) \} \\
\mathcal{D}(5,8) &= \{ \left(\begin{smallmatrix} 5 & 0 \\ 0 & 8 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 2 \\ 2 & 8 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 1 \\ 1 & 5 \end{smallmatrix}\right) \} \cup \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right) \}
\end{aligned}$$

- Since $\mathcal{D}(4,5) \cap \mathcal{D}(5,5) = \{ \left(\begin{smallmatrix} 4 & 2 \\ 2 & 5 \end{smallmatrix}\right) \}$ we can identify the corresponding curve. Hence we also know $\left(\begin{smallmatrix} 5 & 1 \\ 1 & 5 \end{smallmatrix}\right)$ (and $\left(\begin{smallmatrix} 4 & 0 \\ 0 & 5 \end{smallmatrix}\right)$).
- Similarly $\left(\begin{smallmatrix} 5 & 2 \\ 2 & 8 \end{smallmatrix}\right)$ can be matched by $\mathcal{D}(5,8) \cap \mathcal{D}(5,9) = \{ \left(\begin{smallmatrix} 5 & 2 \\ 2 & 8 \end{smallmatrix}\right) \}$.

In the end we find that:

$$\begin{aligned} \mathcal{C}_{\left(\begin{smallmatrix} 5 & 2 \\ 2 & 8 \end{smallmatrix}\right)} &: \text{ the genus 1 curve} \\ \mathcal{C}_{\left(\begin{smallmatrix} 5 & 1 \\ 1 & 5 \end{smallmatrix}\right)} &: \text{ the genus 3 hyperelliptic curve} \\ \mathcal{C}_{\left(\begin{smallmatrix} 5 & 0 \\ 0 & 8 \end{smallmatrix}\right)} &: \text{ the genus 3 non-hyperelliptic curve} \end{aligned}$$

Thanks for listening!