Computing "isogeny graphs" using CM lattices

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Motivation for computing isogenies

- Point counting.
- Computing CM invariants.
- Endomorphism ring computations.
- Transporting discrete log problems.
- Visiting Corsica...

History of isogenies

Genus 1, we have Vélu's formulae.

Genus 2, from a computational perspective we had:

- Richelot $(2,2)$-isogenies
- $(3,3)$-isogenies (Carls-Kohel-Lubicz, Bröker-G.-Lauter)

And, as the CHIC project has progressed:

$\rightarrow$ (l^2, l^2)-isogenies for $l \lesssim 40$ (Lubicz-Robert)

$\Rightarrow$ (l,l)-isogenies for $l \lesssim 1000$ now possible using the Magma package `AVIsogenies` (Bisson-Cosset-Robert)

Other types of isogenies:

- Explicit endomorphisms for RM families:
  - $\sqrt{2}$ in genus 2 (Bending, Gaudry, Mestre,...)
  - $\frac{1+\sqrt{5}}{2}$ in genus 2 (Kohel-Smith, Takashima,...)
  - $\zeta_{2g+1} + \zeta_{2g+1}^{-1}$ in genus g (Mestre, Smith, Tautz-Top-Verberkmoes,...)
- $(2, 2, 2)$-isogenies for generic genus 3 curves (Lehavi-Ritzenthaler)
- Ben Smith can tell you more and will undoubtedly find more!..

Lattices of elliptic curves

isomorphisms

Let E be an elliptic curve over $\mathbb{C}$.

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda, \text{ where } \Lambda = \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} \subset \mathbb{C} \text{ is a lattice.}$$

In particular, $\{\alpha_1, \alpha_2\} \subset \mathbb{C}$ are $\mathbb{R}$-linearly independent, so one of

$$(\alpha_1/\alpha_2)^{\pm 1} \in \mathbb{H}_1 := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

Order our basis $\langle \alpha_1, \alpha_2 \rangle$ so that $\alpha_1/\alpha_2 \in \mathbb{H}_1$.

The set of lattices Λ_0 with ordered bases such that $\mathbb{C}/\Lambda_0 \cong E(\mathbb{C})$ is given by the orbit

$$\mathbb{C}^* \backslash \Lambda / \mathrm{SL}_2(\mathbb{Z})$$

- Left action : rescale basis by $\lambda \in \mathbb{C}^*$
  $\lambda \cdot \langle \alpha_1, \alpha_2 \rangle = \langle \lambda\alpha_1, \lambda\alpha_2 \rangle$
- Right action: change basis by $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$
  $\langle \alpha_1, \alpha_2 \rangle \cdot M = \langle a\alpha_1 + b\alpha_2, c\alpha_1 + d\alpha_2 \rangle$

In particular, $\langle \tau, 1 \rangle \cdot M \cong \langle M \cdot \tau, 1 \rangle$ where

$$M \cdot \tau := \frac{a\tau + b}{c\tau + d}$$

From this, we see the usual $\mathrm{SL}_2(\mathbb{Z})$-action on the upper half plane.

Lattices of elliptic curves

isogenies

- An isogeny $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is induced by a $\mathbb{C}$-linear map $\varphi : \mathbb{C} \to \mathbb{C}$ with $\varphi\Lambda \subseteq \Lambda'$.
- Fixing a basis for Λ' we can represent this by $R_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z})$ with $ad - bc = n > 0$.
- The degree/kernel of the isogeny is given by the elementary divisors of R_φ.
- $n = 1 \implies R_\varphi \in \mathrm{SL}_2(\mathbb{Z})$ is an isomorphism, as expected.

Lattices of elliptic curves

"Isogeny graphs"

Usual definition of a T-isogeny graph:

- Vertices: isomorphism classes of elliptic curves
- Edges: isogenies of type T

"Equivalent" definition:

- Vertices: lattices upto homothety
- Edges: T-isogenies between lattices

Example: 2-isogeny graphs

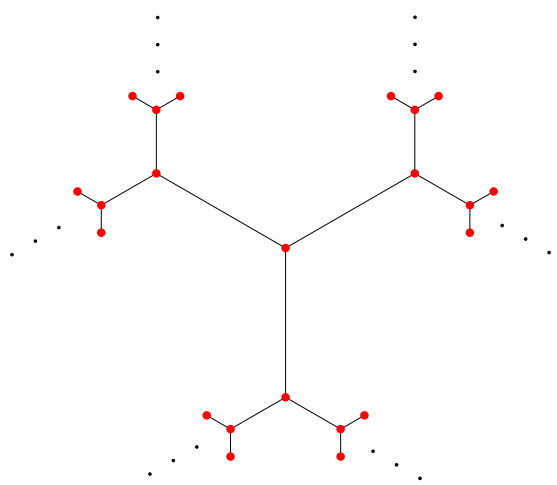
For l prime, there are $l+1$ cyclic l-isogenies

$$\mathcal{R}_l = \left\{ \left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right) x \mid x \in \mathrm{SL}_2(\mathbb{Z})/\Gamma_0(l) \right\}$$

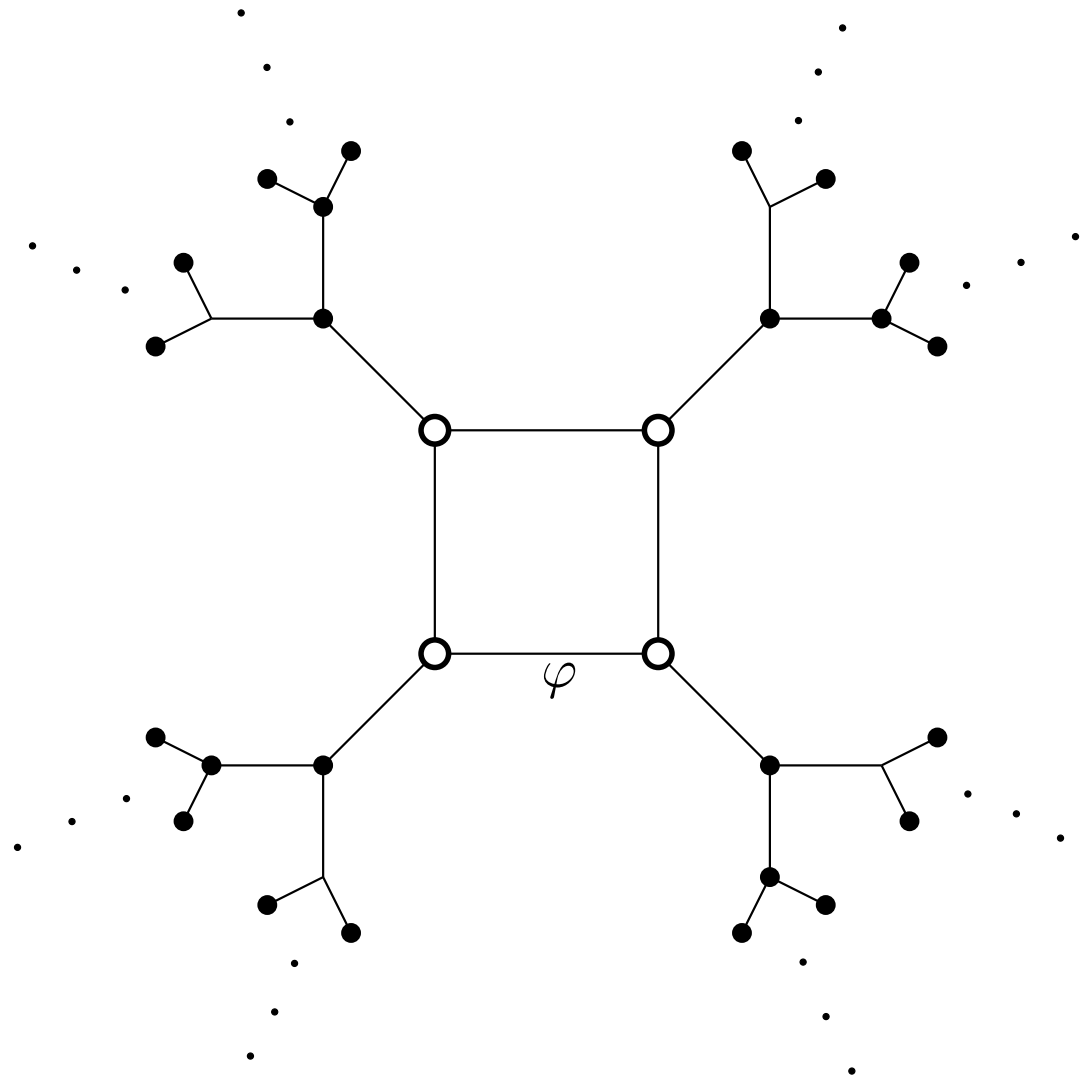
where $\Gamma_0(l) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{l} \right\}$
Up to isomorphism, the 2-isogenies can be represented by:

$$\mathcal{R}_2 = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

For a generic $\tau \in \mathbb{H}_1$ (the case where $\mathrm{End}(\langle \tau, 1 \rangle) \cong \mathbb{Z}$) the isogeny graph is a 3-ary tree



For a CM point $\tau \in \mathbb{H}_1$ (the case where $\mathrm{End}(\langle \tau, 1 \rangle) \cong \mathcal{O}$ is an order in $\mathbb{Q}(\sqrt{-D})$) the isogeny graph is determined by $\mathrm{Pic}(\mathcal{O})$



$\ker \varphi = \mathfrak{a}$, $[\mathfrak{a}^4] = [\mathcal{O}] \in \mathrm{Pic}(\mathcal{O})$

Lattices of abelian surfaces

isomorphisms

Let A be a principally polarized (PP) abelian surface over $\mathbb{C}$.

$$A(\mathbb{C}) \cong \mathbb{C}^2/\Lambda, \text{ where } \Lambda \cong \mathbb{Z}^4 \text{ is a PP lattice}$$

Such a lattice comes equipped with a symplectic basis (wrt the polarization). Using this basis we can then write $\Lambda = \Pi\mathbb{Z}^4$ where $\Pi = \langle \Pi_1 \ \Pi_2 \rangle \in \mathrm{Mat}(2 \times 4, \mathbb{C})$ called the period matrix. This matrix satisfies the *Riemann relations*:

$$\Pi_2 {}^t\Pi_1 - \Pi_1 {}^t\Pi_2 = 0 \qquad \text{(RR1)}$$

$$i(\Pi_2 {}^t\overline{\Pi}_1 - \Pi_1 {}^t\overline{\Pi}_2) > 0 \qquad \text{(RR2)}$$

The set of PP lattices Λ_0 for which $\mathbb{C}/\Lambda_0 \cong A(\mathbb{C})$ as PPAS's is given by the orbit

$$\mathrm{GL}_2(\mathbb{C})\backslash\Lambda/\mathrm{Sp}_4(\mathbb{Z})$$

- Left action : $\lambda \in \mathrm{GL}_2(\mathbb{C})$ sends Π to $\lambda\Pi$
- Right action: $M \in \mathrm{Sp}_4(\mathbb{Z})$ sends Π to $\Pi\, {}^tM$

(RR $\Rightarrow$) each orbit has a representative of the form $\langle \tau \ I \rangle$ where

$$\tau \in \mathbb{H}_2 := \left\{ Z \in \mathbb{M}_2(\mathbb{C}) \mid {}^tZ = Z \text{ and } \ \mathrm{Im}\, Z > 0 \right\}$$

From this we derive the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z})$ on $\tau \in \mathbb{H}_2$:

$$M \cdot \tau := (a\tau + b)(c\tau + d)^{-1}$$

Lattices of abelian surfaces

Isogenies

Let $A = (\mathbb{C}^2/\Lambda, \chi)$ be a polarized abelian surface over $\mathbb{C}$.

- An isogeny $\varphi : \mathbb{C}^2/\Lambda \to \mathbb{C}^2/\Lambda'$ induces a $\mathbb{C}$-linear map $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ with $\varphi\Lambda \subseteq \Lambda'$.
- Fixing a basis of Λ' we can represent this by $R_\varphi \in \mathbb{M}_4(\mathbb{Z})$, with $\deg \varphi = \det R_\varphi = n > 0$.

In fact,

$$\varphi : (A, \chi) \to C/\Lambda' = (A', \chi')$$

is an isogeny of PAS's, but not necessarily polarization preserving. (In general $\chi \neq \varphi^* \chi'$)

(l,l)-isogenies

- Isogenies $\varphi : A \to A'$ for which $\ker\varphi \cong (\mathbb{Z}/l\mathbb{Z})^2$ is a maximal Weil-isotropic l-subgroup of $A[l]$ preserve the polarization class and are called (l,l)-isogenies.
- For l prime there are

$$l^3 + l^2 + l + 1$$

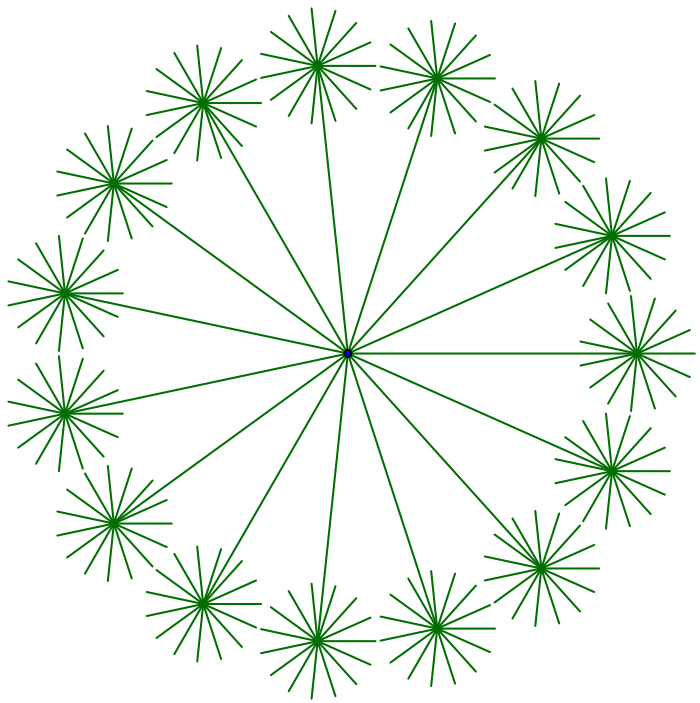
(l,l)-isogenies up to isomorphism, represented by:

$$\mathcal{R}_{l,l}^{(2)} = \left\{ \mathrm{diag}(l,l,0,0)x \mid x \in \mathrm{Sp}_4(\mathbb{Z})/\Gamma_0^{(2)}(l) \right\}$$

where $\Gamma_0^{(2)}(l) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z}) \mid c \equiv 0 \pmod{l} \}$.

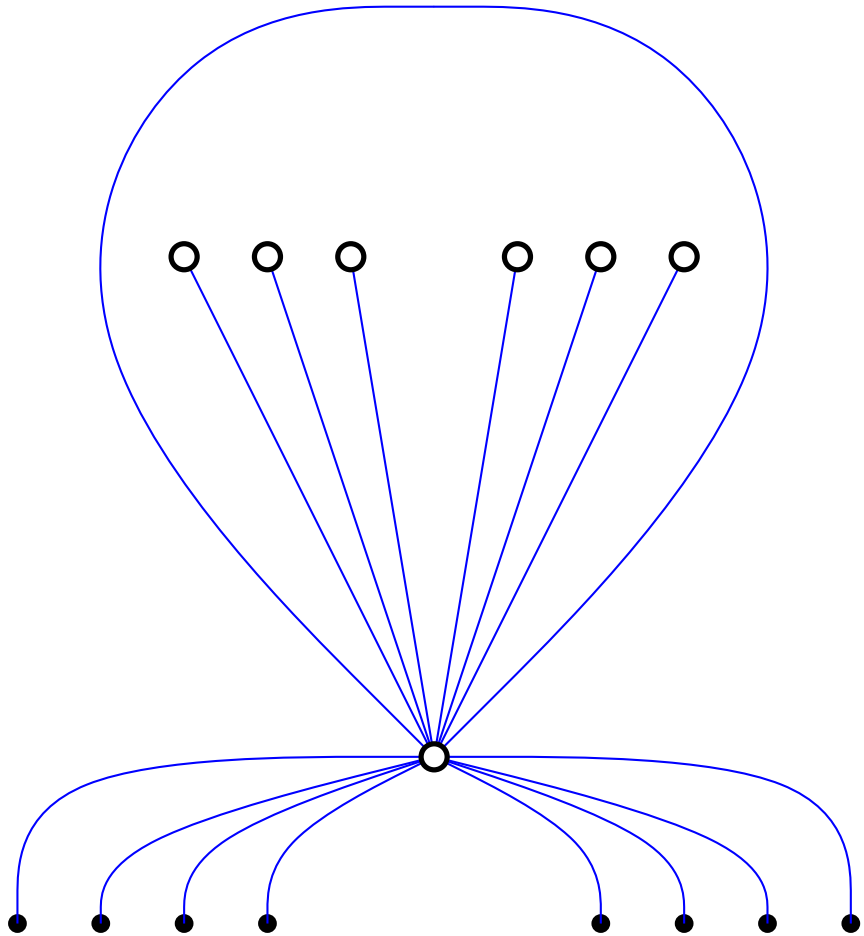
Generic example

For a generic $\tau \in \mathbb{H}_2$ (the case where $\mathrm{End}(\langle \tau, I \rangle) \cong \mathbb{Z}$) the $(2,2)$-isogeny graph is a 15-ary tree



RM example - squiddy the 6-eyed octopus

Here's part of the $(2,2)$ isogeny graph of an RM lattice:
$\mathrm{End}(\langle \tau, I \rangle) = \mathbb{Z}[\sqrt{2}]$



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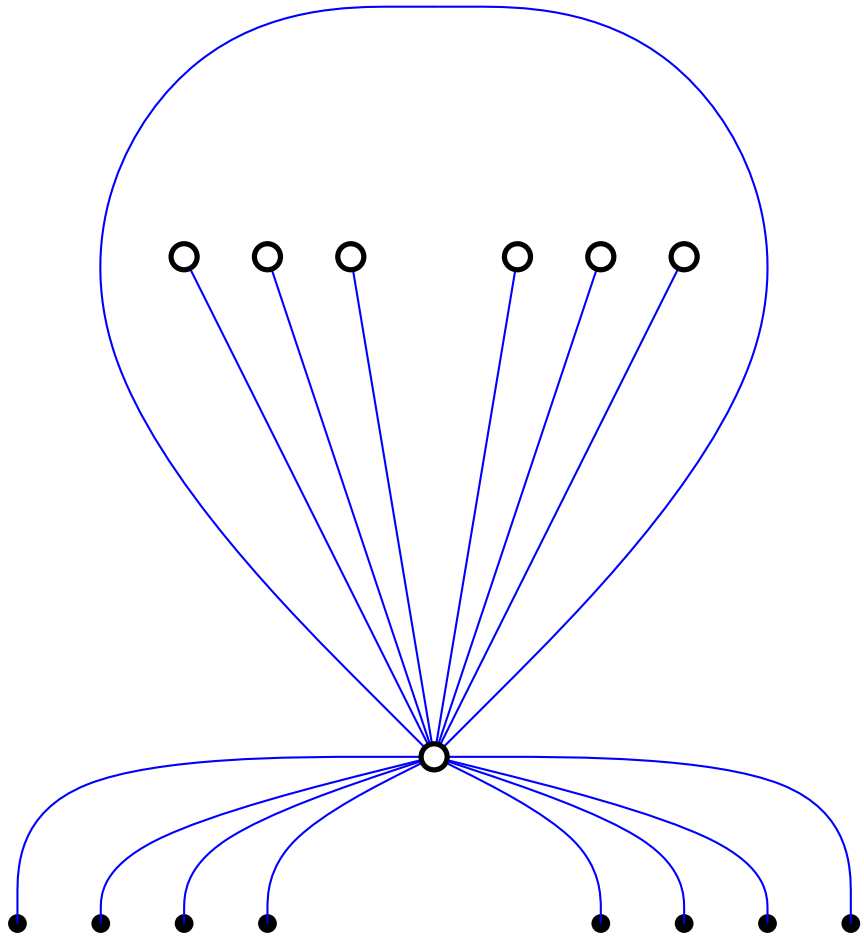
$\mathrm{End}(\langle \tau, I \rangle) = \mathbb{Z}[\sqrt{2}] = \mathrm{Jac}(C)$ where
$C: \ y^2 = x^6 - 4x^5 - 12x^4 + 2x^3 + 8x^2 + 8x - 7$ over $\mathbb{C}$



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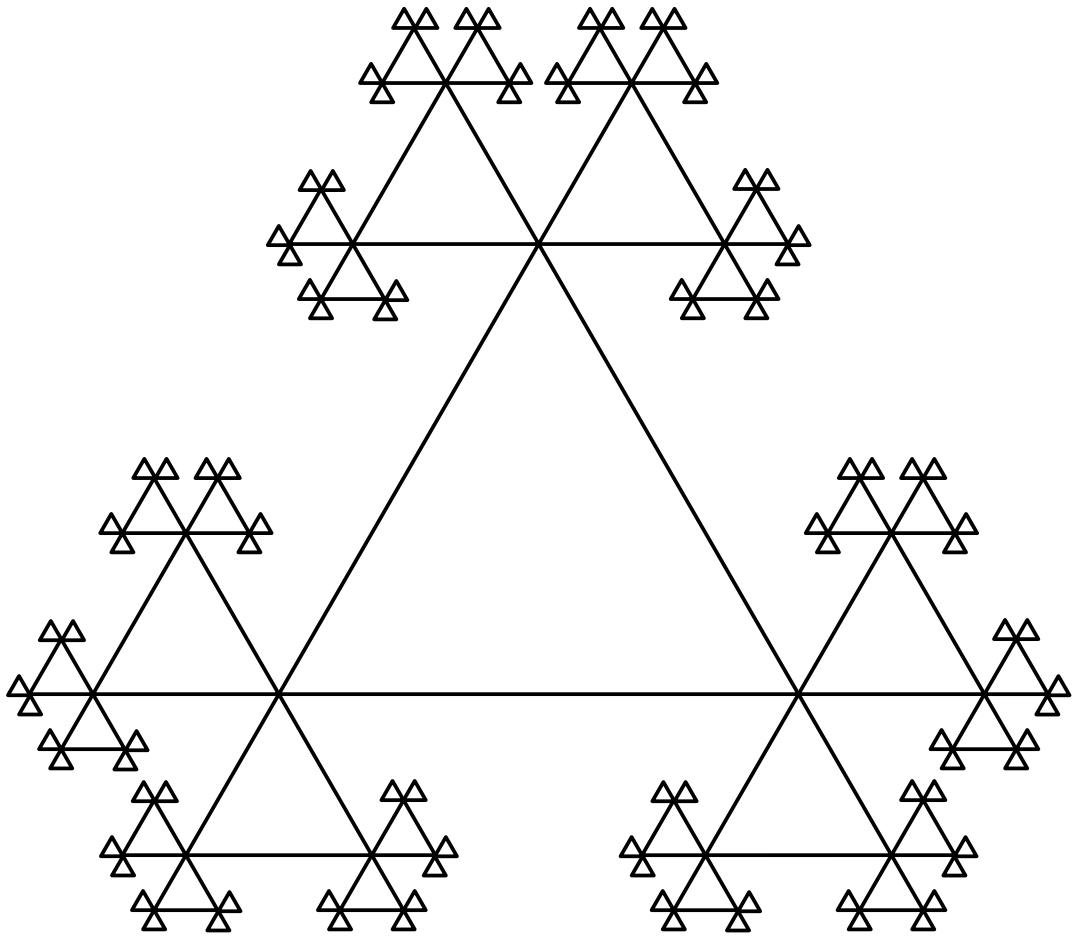
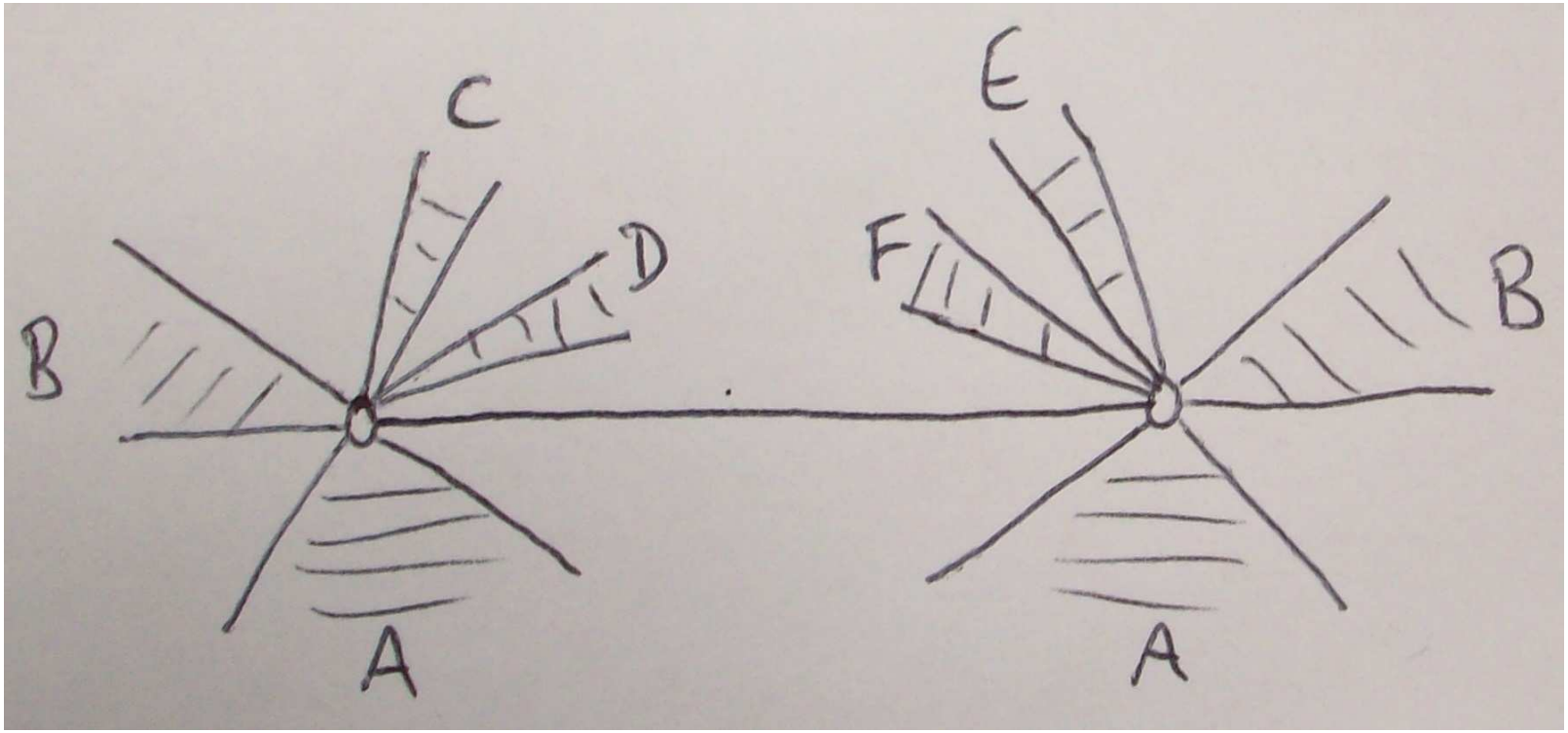


part of graph with only (some) maximal RM points shown

CM example: a $(3,3)$-isogeny graph

$K = \mathbb{Q}[X]/(X^4 + 12X^2 + 18)$, cyclic Galois group, class number 2. $3\mathcal{O}_K = \mathfrak{p}^2 \Rightarrow$ the two CM lattices with $\mathcal{O}_K$-multiplication are connected by a $(3,3)$-isogeny.



$A_{27}, B_9, C_9, D_9, E_9, F_9$ are the endomorphism rings of the nonmaximal "leaf" vertices (appearing with multiplicities 18,9,6,6,6,6 resp.) $18 + 9 + 6 + 6 + 1 = 40$ isogenous points.

Connection to isogeny graphs over finite fields

An ordinary PPAS over $\mathbb{F}_q$ is the reduction of an abelian surface over $\mathbb{C}$ having CM by an order in $K = \mathbb{Q}(\pi)$ where π is an ordinary Weil number of norm q^2.
To obtain an (l, l)-isogeny graph for PPAS's over $\mathbb{F}_q$ in the isogeny class given by $\pi \in K$, do the following:

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1. Take a principally polarizable ideal class of $\mathcal{O}_K$: $(\mathfrak{a}, \xi)$ where $\xi \in K$ is purely imaginary and $\xi \mathfrak{a}\overline{\mathfrak{a}} = \mathfrak{D}_{K/\mathbb{Q}}^{-1}$, the inverse different. This is our starting point.

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2. Compute a Frobenius basis $\Pi\mathbb{Z}^4$ for the rank four $\mathbb{Z}$-module $\mathfrak{a}$ with respect to ξ; the symplectic form is

$$E\colon (x, y) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(\xi\overline{x}y)$$

and we want the matrix of E to be $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

3. Compute (l, l)-isogenous images:

   For each isogeny transformation $M \in \mathcal{R}_{l,l}^{(2)}$:

   - Compute the isogenous period matrix

$$\Pi' = \Pi\, {}^tM.$$

   The principally polarized CM lattice is $(\mathfrak{a}', \xi') = (\Pi'\mathbb{Z}^4, l^{-1}\xi)$.

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   - Test whether

     $$\pi\mathfrak{a}' \subset \mathfrak{a}'.$$

     If true, this means that $\pi, \overline{\pi} \in \mathrm{End}(\mathfrak{a}')$ and that the reduction of this isogenous PPAS $\mathbb{C}^2/\Pi'\mathbb{Z}^4$ is defined over $\mathbb{F}_q$.
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Recursively run algorithm on unexplored isomorphism classes of CM lattices. Isomorphism test:

$$(\mathfrak{a}, \xi) \cong (\mathfrak{a}', \xi') \text{ iff } \begin{cases} \mathfrak{a}' = \gamma\mathfrak{a} \\ \xi' = (\gamma\overline{\gamma})^{-1}\xi \end{cases} \quad (\exists\, \gamma \in K)$$

Disadvantages:

- We can compute complex approximations of absolute invariants for CM lattices, but constructing CM moduli over $\mathbb{F}_q$ seems rather intractable.
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Advantages:

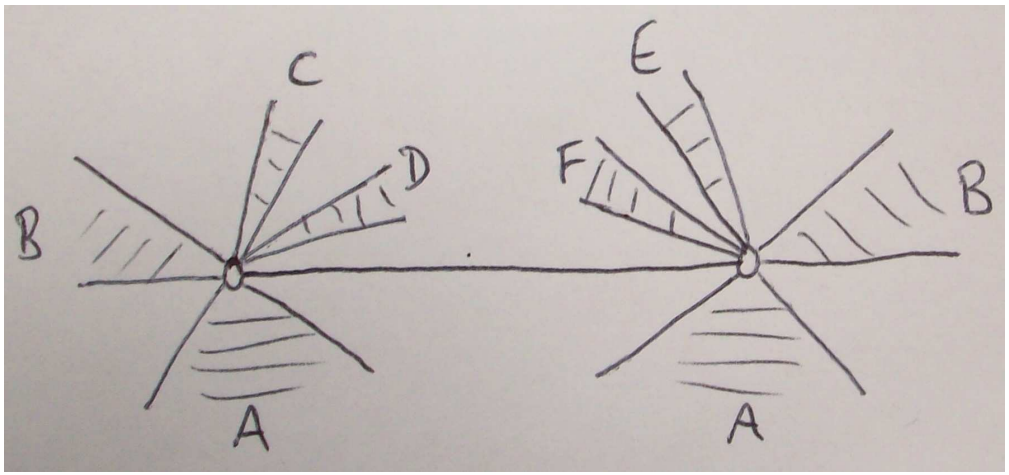
- Endomorphism rings of CM lattices are easy to compute (= multiplier ring of lattice)

Generalisations:

- Use a different set $\mathcal{R}$ of isogeny transformations, not necessarily always polarization preserving (e.g. cyclic l-isogenies).
- Higher genus.

Example 1: our old friend $K = \mathbb{Q}[X]/(X^4 + 12X^2 + 18)$

$\mathrm{Gal}(K/\mathbb{Q}) = C_4 = \langle\sigma\rangle$ and $\mathrm{Cl}(\mathcal{O}_K) = \mathbb{Z}/2\mathbb{Z}$.
Let $q = 127$. We have $K = \mathbb{Q}(\pi)$ where
$\pi^4 + 28\pi^3 + 378\pi^2 + 28q\pi + q^2 = 0$ is an ordinary Weil number.



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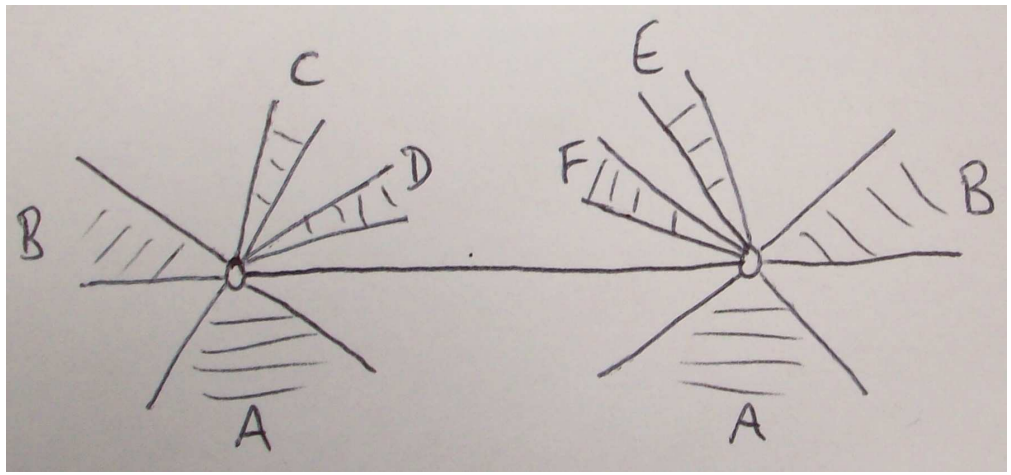

$A_{27}, B_9, C_9, D_9, E_9, F_9$ are the endomorphism rings of the nonmaximal "leaf" vertices (appearing with multiplicities 18,9,6,6,6,6 resp.) Of the six proper suborders, only F_9 contains π.

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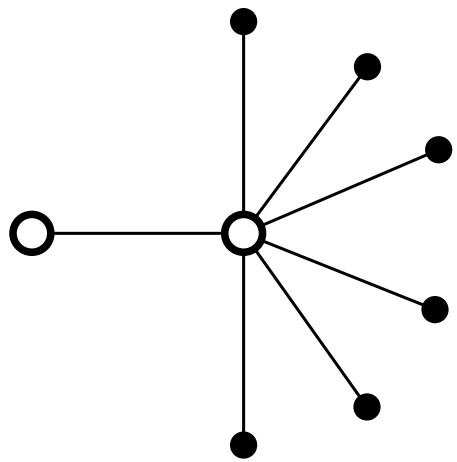

Lesson: graph structure alone is not sufficient information to determine the endomorphism ring.

Example 2: $K = \mathbb{Q}[X]/(X^4 + 22X + 73)$

$(3,3)$-isogeny graph over $\mathbb{F}_{1609}$

$\mathrm{Gal}(K/\mathbb{Q}) \cong$ dihedral group of order 8.
Let $q = 1609$. We have $K = \mathbb{Q}(\pi) = \mathbb{Q}(\psi)$ where
$\pi^4 + 76\pi^3 + 2934\pi^2 + 76q\pi + q^2 = 0$ and
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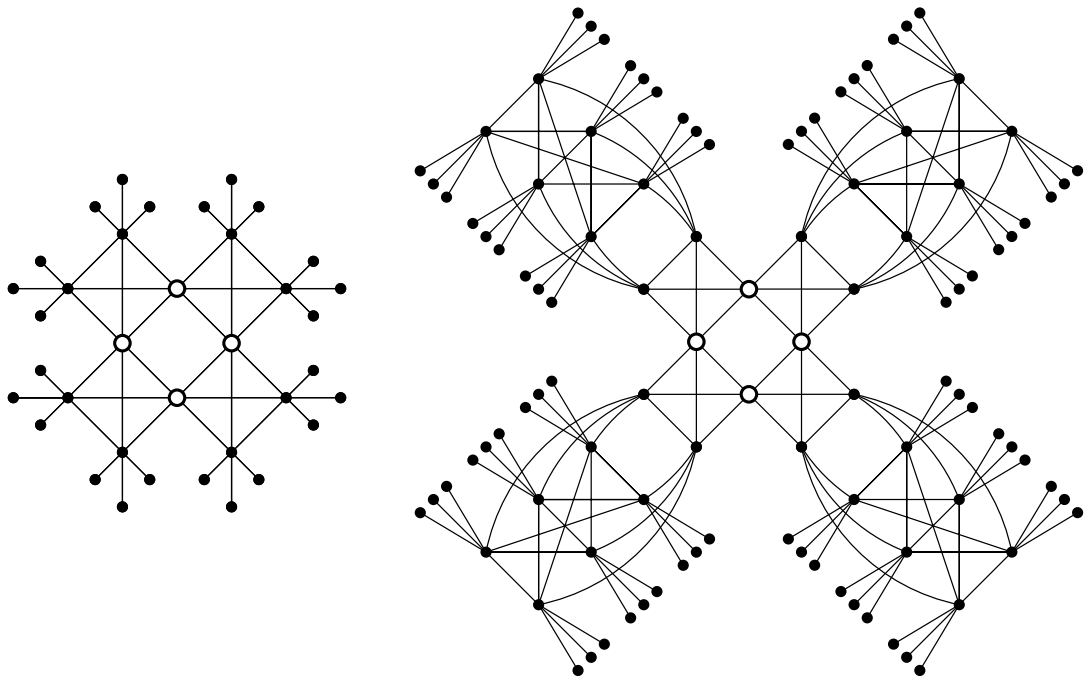
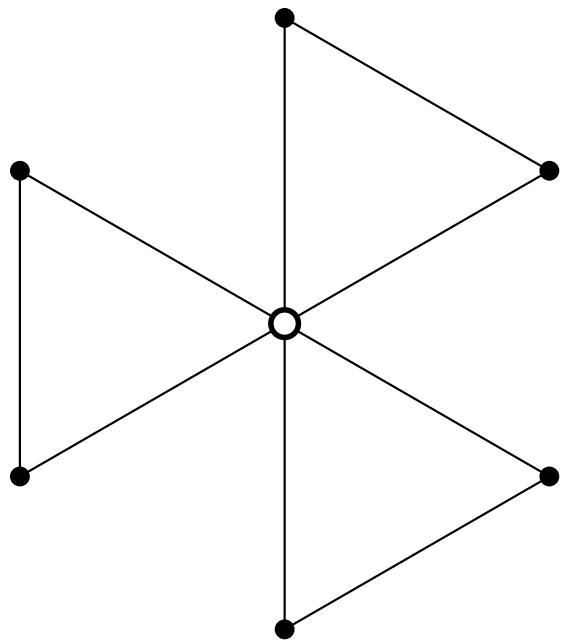


Lesson: we can have cycles involving nonmaximal points (invertible ideals of norm l^2 can exist in non l-maximal orders).

Example 3: $K = \mathbb{Q}[X]/(X^4 + 598X^2 + 70969)$

$(1, 2)$-isogenies over $\mathbb{F}_{3^7}$

Let $q = 3^7$. We have $K = \mathbb{Q}(\pi)$ where $\pi^4 + 124\pi^3 + 7418\pi^2 + 124q\pi + q^2 = 0$ is an ordinary Weil number.
$(2, 2)$-isogeny graph:



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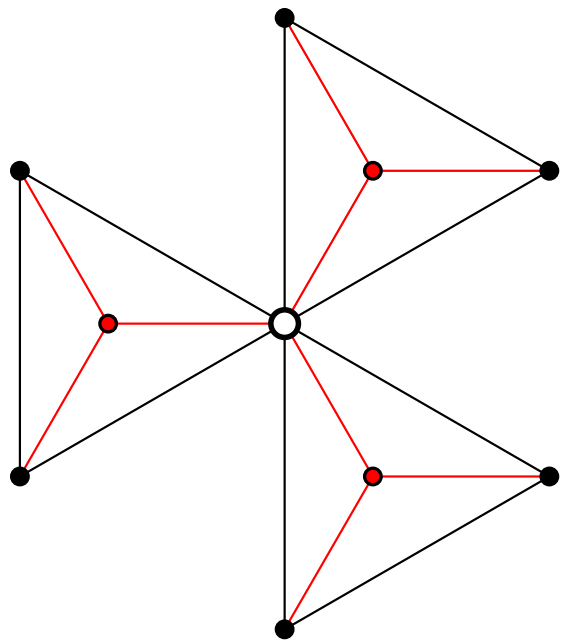

The end

Genus 3 isogeny graphs?

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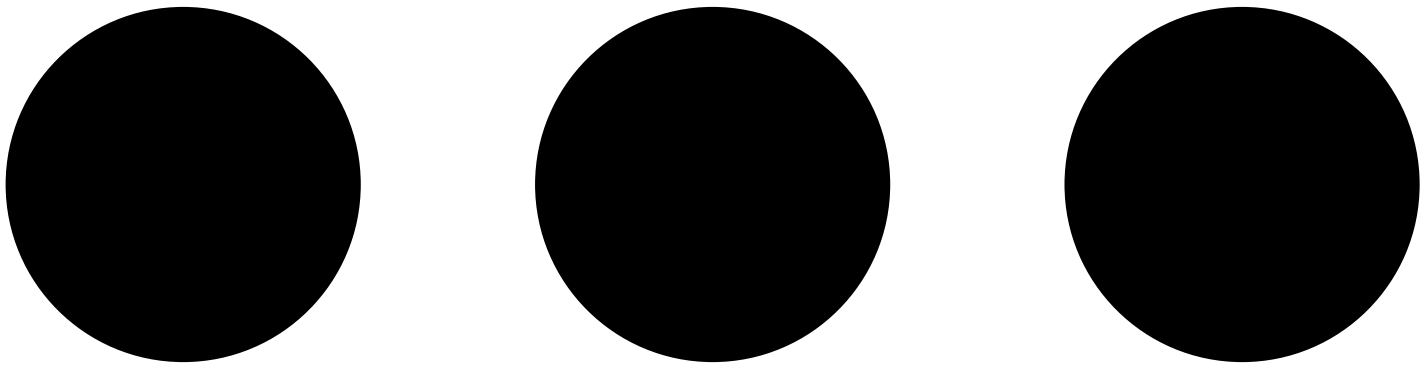

Thanks for your attention.