1.7 The Radical of an Ideal

Let A be a ring, and consider $X \subseteq A$.

Define the **radical of** X (**with respect to** A) to be

$$r(X) \;=\; \{\, z \in A \mid z^n \in X \quad \exists n \geq 1 \,\} \, .$$

— comprising all "nth roots" of elements of X for all positive n.

Clearly

$$\boxed{r\Big(\bigcup_\alpha X_\alpha\Big) \;=\; \bigcup_\alpha r(X_\alpha)}$$

for any family of subsets X_α of A .

Proposition: The set of zero-divisors of A is equal to its own radical which is

$$\bigcup_{x \neq 0} r(\mathrm{Ann}\, x) \, .$$

Proof: Put $D = \{$ zero-divisors of $A\}$. Then

$$D = \bigcup_{x \neq 0} \mathrm{Ann}\,(x) \; .$$

Certainly

$$D \subseteq r(D) \; .$$

Suppose $y \in r(D)$, so

$$y^k \in D \qquad (\exists k \geq 1) \; ,$$

so

$$y^k x = 0 \qquad (\exists x \neq 0) \; .$$

If $k = 1$ then $y \in D$.

If $k > 1$ then

$$y(y^{k-1}x) \quad = \quad 0$$

so either $y^{k-1}x \neq 0$, whence $y \in D$, or

$$y^{k-1}x \quad = \quad 0 \ ,$$

whence $y \in D$ by an inductive hypothesis.

Thus

$$D = r(D) = r\Big(\bigcup_{x\neq 0} \mathrm{Ann}\,(x)\Big)$$

$$= \bigcup_{x\neq 0} r(\mathrm{Ann}\,(x))\,,$$

and the Proposition is proved.

Now suppose $I \lhd A$.

Then

$$r(I) = \{ x \in A \mid x^n \in I \qquad \exists n \in \mathbb{Z}^+ \}$$

$$= \{ x \in A \mid I + x^n = I \qquad \exists n \in \mathbb{Z}^+ \}$$

$$= \{ x \in A \mid (I + x)^n = I \qquad \exists n \in \mathbb{Z}^+ \}$$

so that

$$r(I) \;=\; \phi^{-1}\big(N_{A/I}\big) \;\lhd\; A$$

where $\phi : A \rightarrow A/I$ is the natural map and $N_{A/I}$ denotes the nilradical of A/I .

The radical of an ideal I of A is the preimage under the natural map of the nilradical of A/I .

Exercises: Let I , J be ideals of A . Verify the following:

(1) $r(I) \supseteq I$;

(2) $r(r(I)) = r(I)$;

(3) $r(IJ) = r(I \cap J) = r(I) \cap r(J)$;

(4) $r(I) = A \iff I = A$;

(5) $r(I + J) = r(r(I) + r(J))$.

Exercise: If P is a prime ideal of A then

$$(\forall n \in \mathbb{Z}^+) \qquad r(P^n) \;=\; P\,.$$

Example: Let $A = \mathbb{Z}$ and $I = m\mathbb{Z}$ where $m \geq 2$. Write

$$m \;=\; p_1^{\alpha_1} \ldots p_r^{\alpha_r}$$

for distinct primes $p_1, \ldots, p_r$ and positive integers $\alpha_1, \ldots, \alpha_r$.

Observe that

$$(p_1 \ldots p_r)^\beta \;\in\; I$$

where

$$\beta \;=\; \max\{\,\alpha_1, \ldots, \alpha_{\mathrm{r}}\,\}$$

so

$$p_1 \ldots p_r \;\in\; r(I)\;,$$

so

$$p_1 \ldots p_r\,\mathbb{Z} \;\;\subseteq\;\; r(I)\;.$$

On the other hand, if $z \in r(I)$ then some positive power of z is divisible by m , from which it follows that z is divisible by $p_1 \dots p_r$. Thus

$$r(I) \;=\; p_1 \dots p_r \, \mathbb{Z} \; .$$

Notice that

$$r(I) \;=\; \bigcap_{i=1}^{r} p_i \mathbb{Z} \; ,$$

the intersection of all prime ideals containing I .

This illustrates a general phenomenon:

Theorem: The radical of an ideal is the intersection of the prime ideals containing it.

Proof: Let $I \lhd A$. Then

$$r(I) \;=\; \phi^{-1}\big(N_{A/I}\big)$$

where $N_{A/I}$ is the nilradical of A/I , and ϕ is the natural map.

By an earlier Theorem,

$N_{A/I}$ is the intersection of all prime ideals of A/I .

But

Easy Exercise: any prime ideal of A/I has the form P/I where P is a prime ideal of A containing I .

Thus

$$
\begin{aligned}
r(I) &= \phi^{-1}\Bigg(\bigcap_{\text{prime ideals } P \supseteq I} P/I \Bigg) \\
&= \bigcap_{\text{prime ideals } P \supseteq I} \phi^{-1}(P/I) \\
&= \bigcap_{\text{prime ideals } P \supseteq I} P \, .
\end{aligned}
$$

Proposition: Suppose $I, J \lhd A$ such that $r(I)$ and $r(J)$ are coprime.

Then I and J are coprime.

Proof: By earlier exercises,

$$r(I+J) \;=\; r(r(I)+r(J)) \;=\; r(A) \;=\; A$$

so that $I+J = A$, and the Proposition is proved.