1.1 Rings and Ideals

A **ring** A is a set with $+$, $\bullet$ such that

(1) $(A, +)$ is an abelian group;

(2) $(A, \bullet)$ is a semigroup;

(3) $\bullet$ distributes over $+$ on both sides.

In this course all rings A are **commutative**, that is,

(4) $(\forall x, y \in A) \quad x \bullet y = y \bullet x$

and have an **identity element** 1 (easily seen to be unique)

(5) $(\exists 1 \in A)(\forall x \in A) \quad 1 \bullet x = x \bullet 1 = x$.

If $1 = 0$ then $A = \{0\}$ (easy to see), called the **zero ring**.

Multiplication will be denoted by juxtaposition, and simple facts used without comment, such as

$(\forall x, y \in A)$

$$x\,0 = 0\,,$$

$$(-x)y = x(-y) = -(xy)\,,$$

$$(-x)(-y) = xy\,.$$

Call a subset S of a ring A a **subring** if

(i)	$1 \in S$;	
(ii)	$(\forall x, y \in S)$	$x + y\,,\ xy\,,\ -x\ \in S$.

Condition (ii) is easily seen to be equivalent to

(ii)′	$(\forall x, y \in S)$	$x - y\,,\ xy\ \in S$.

Note: In other contexts authors replace the condition $1 \in S$ by $S \neq \emptyset$ (which is not equivalent!).

Examples:

(1) $\mathbb{Z}$ is the only subring of $\mathbb{Z}$.

(2) $\mathbb{Z}$ is a subring of $\mathbb{Q}$, which is a subring of $\mathbb{R}$, which is a subring of $\mathbb{C}$.

(3) $\mathbb{Z}[i] \;=\; \{\, a + bi \mid a, b \in \mathbb{Z} \,\} \quad (i = \sqrt{-1})$,

the ring of **Gaussian integers** is a subring of $\mathbb{C}$.

(4) $\mathbb{Z}_n \;=\; \{\, 0, 1, \ldots, n-1 \,\}$

with addition and multiplication mod n .

(Alternatively $\mathbb{Z}_n$ may be defined to be the **quotient ring** $\mathbb{Z}/n\mathbb{Z}$, defined below).

(5) R any ring, x an indeterminate. Put

$$R[[x]] = \{a_0 + a_1 x + a_2 x^2 + \ldots \quad | \; a_0, a_1, \ldots \in R \} ,$$

the set of **formal power series over** R , which becomes a ring under addition and multiplication of power series. Important subring:

$$R[x] = \{a_0 + a_1 x + \ldots + a_n x^n \quad | \quad n \geq 0 , \; a_0, a_1, \ldots, a_n \in R \} ,$$

the ring of **polynomials over** R .

Call a mapping $f : A \to B$ (where A and B are rings) a **ring homomorphism** if

(a) $f(1) = 1$;

(b) $(\forall x, y \in A)$

$$f(x + y) = f(x) + f(y)$$

and

$$f(xy) = f(x)f(y) ,$$

in which case the following are easily checked:

(i) $f(0) = 0$;

(ii) $(\forall x \in A) \quad f(-x) = -f(x)$;

(iii) $f(A) = \{f(x) \mid x \in A\}$, the **image** of f is a subring of B ;

(iv) Composites of ring hom's are ring hom's.

An **isomorphism** is a bijective homomorphism, say $f : A \to B$, in which case we write

$$A \;\cong\; B \qquad \text{or} \qquad f : A \cong B \; .$$

It is easy to check that

$\cong$ is an equivalence relation.

A nonempty subset I of a ring A is called an **ideal**, written $I \lhd A$, if

(i) $(\forall x, y \in I) \quad x + y\,,\ -x \in I$

[clearly equivalent to

(i)′ $(\forall x, y \in I) \quad x - y \in I$];

(ii) $(\forall x \in I)(\forall y \in A) \quad xy \in I$.

In particular I is an additive subgroup of A, so we can form the quotient group

$$A/I \;=\; \{\, I + a \mid a \in A \,\}\,,$$

the group of **cosets** of I,

with addition defined by, for $a, b \in A$,

$$(I + a) \;+\; (I + b) \;=\; I + (a + b)\,.$$

Further A/I forms a ring by defining, for $a, b \in A$,

$$(I + a)\,(I + b) \;=\; I \;+\; (a\,b)\;.$$

Verification of the ring axioms is straightforward.

— only tricky bit is first checking multiplication is well-defined:

If $I + a = I + a'$ and $I + b = I + b'$ then

$$a - a' \, , \;\; b - b' \in I \, ,$$

so

$$ab - a'b' = ab - ab' + ab' - a'b'$$

$$= a(b - b') + (a - a')b' \in I \, ,$$

yielding $I + ab = I + a'b'$.

We call A/I a **quotient ring**.

The mapping

$$\phi : A \to A/I \ , \quad x \mapsto I + x$$

is clearly a surjective ring homomorphism, called the **natural map**, whose kernel is

$$\ker \phi \ = \ \{ \, x \in A \ | \ I + x \ = \ I \, \} \ = \ I \ .$$

Thus all ideals are kernels of ring homomorphisms. The converse is easy to check, so kernels of ring homomorphisms with domain A are precisely ideals of A .

The following important result is easy to verify:

Fundamental Homomorphism Theorem:

If $f : A \to B$ is a ring homomorphism with kernel I and image C then

$$A/I \;\cong\; C \; .$$

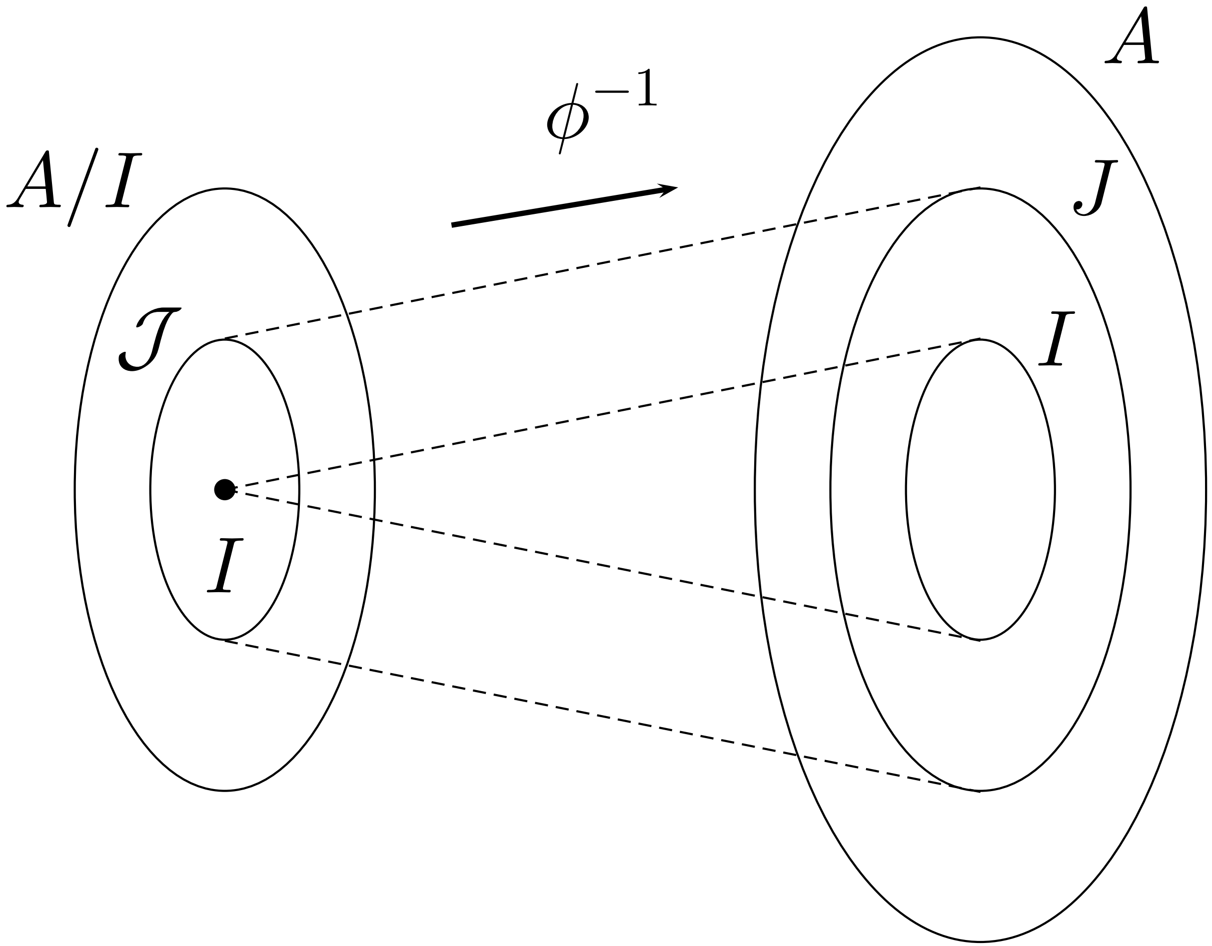
Proposition: Let $I \lhd A$ and $\phi : A \to A/I$ be the natural map. Then

(i) ideals $\mathcal{J}$ of A/I have the form

$$\mathcal{J} \ = \ J/I \ = \ \{\, I + j \ \mid \ j \in J \,\}$$

for some J such that $I \subseteq J \lhd A$;

(ii) ϕ^{-1} is an inclusion-preserving bijection between ideals of A/I and ideals of A containing I .


ϕ^{-1}
A
A/I
J
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I

Example: The ring

$$\mathbb{Z}_n \;=\; \{\, 0, 1, \ldots, n-1 \,\}$$

with mod n arithmetic is isomorphic to $\mathbb{Z}/n\mathbb{Z}$:

follows from the Fundamental Homomorphism Theorem, by observing that the mapping $f : \mathbb{Z} \rightarrow \mathbb{Z}_n$ where

$$f(z) \;=\; \text{remainder after dividing } z \text{ by } n$$

is a ring homomorphism with image $\mathbb{Z}_n$ and kernel $n\mathbb{Z}$.

Example: $\mathbb{Z}/9\mathbb{Z} \;\cong\; \mathbb{Z}_9$ has ideals

$$\mathbb{Z}/9\mathbb{Z}\,, \quad 3\mathbb{Z}/9\mathbb{Z}\,, \quad 9\mathbb{Z}/9\mathbb{Z}$$

(corresponding under the isomorphism to the ideals $\mathbb{Z}_9$, $\{0,3,6\}$, $\{0\}$ of $\mathbb{Z}_9$)

which correspond under ϕ^{-1} to

$$\mathbb{Z}\,, \quad 3\mathbb{Z}\,, \quad 9\mathbb{Z}$$

respectively, a complete list of ideals of $\mathbb{Z}$ which contain $9\mathbb{Z}$.

Zero-divisors, nilpotent elements and units:

Let A be a ring.

Call $x \in A$ a **zero divisor** if

$$(\exists y \in A) \quad y \neq 0 \quad \text{and} \quad xy = 0 \ .$$

Examples:

2 is a zero divisor in $\mathbb{Z}_{14}$.

$5, 7$ are zero divisors in $\mathbb{Z}_{35}$.

A nonzero ring in which 0 is the only zero divisor is called an **integral domain**.

Examples: $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

We can construct many more because of the following easily verified result:

Proposition: If R is an integral domain then the polynomial ring $R[x]$ is also.

Corollary: If R is an integral domain then the polynomial ring $R[x_1, \ldots, x_n]$ in n commuting indeterminates is also.

Call $x \in A$ **nilpotent** if

$$x^n = 0 \quad \text{for some } n > 0 .$$

All nilpotent elements in a nonzero ring are zero divisors, but not necessarily conversely.

Example: $2 \cdot 3 = 0$ in $\mathbb{Z}_6$, so 2 is a zero divisor, but

$$2^n \quad = \quad \begin{cases} 2 \ \text{ if } n \text{ is odd} \\ \\ 4 \ \text{ if } n \text{ is even} \end{cases}$$

so 2 is not nilpotent in $\mathbb{Z}_6$.

Call $x \in A$ a **unit** if

$$xy = 1 \quad \text{for some } y \in A ,$$

in which case it is easy to see that y is unique, and we write $y = x^{-1}$.

It is routine to check that

the units of A form an abelian group under multiplication.

Examples:

(1) The units of $\mathbb{Z}$ are ± 1 .

(2) The units of $\mathbb{Z}[i]$ are $\pm 1, \pm i$.

(3) If $x \in \mathbb{Z}_n$ then x is a unit iff x and n are coprime as integers. Thus

> all nonzero elements of $\mathbb{Z}_n$ are units iff n is a prime.

A **field** is a nonzero ring in which all nonzero elements are units.

Examples: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_p$, where p is a prime, are fields.

It is easy to check that

all fields are integral domains.

Not all integral domains are fields (e.g. $\mathbb{Z}$).

However integral domains are closely related to fields by the construction of **fields of fractions** described in **Part 3**.

A **principal** ideal P of A is an ideal generated by a single element, that is, for some $x \in A$,

$$P = Ax = xA = \{ ax \mid a \in A \} .$$

Note that

$$A\,1 \;=\; A\,, \quad \text{and} \quad A\,0 \;=\; \{0\}\,.$$

Clearly, for $x \in A$,

$$x \;\text{ is a unit iff }\quad Ax \;=\; A\,.$$

Proposition: Let A be nonzero. TFAE

1. A is a field.
2. The only ideals of A are $\{0\}$ and A.
3. Every homomorphism of A onto a nonzero ring is injective.