Regularisation Effect of Nonlinear Semigroups

Algebra/Geometry/Topology Seminar
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Joint work with Thierry Coulhon (Research University PSL, Paris)

Local diffusion equations

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(1) $\partial_t u - \mathrm{div}(a(x, \nabla u)) + \beta(u) = 0$ in $\Omega \times (0,\infty)$

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$u = u(x,t)$

heat density
density of a biol. population
concentration of a chem. comp.

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infinitesimal change of the density/concentr.

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$$(1) \quad \underbrace{\partial_t u}_{\substack{\text{infinitesimal}\\ \text{change of the}\\ \text{density/concentr.}}} - \underbrace{\mathrm{div}(a(x, \nabla u))}_{\substack{\text{diffusion of the}\\ \text{heat/particles}\\ \text{through } \Omega}} + \beta(u) = 0 \qquad \text{in } \Omega \times (0, \infty)$$

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Local diffusion equations

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(1) $\partial_t u - \mathrm{div}(a(x, \nabla u)) + \beta(u) = 0$ in $\Omega \times (0,\infty)$

$j(x,t) := -a(x, \nabla u)$ indicating "into which direction" & by its length "how much" heat/particles is/are transported.

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$j(x,t) := -a(x,\nabla u)$ indicating "into which direction" & by its length "how much" heat/particles is/are transported.

$$\boxed{\text{A natural assumption: } j(x,t) \;\&\; -\nabla u(x,t) \text{ point into the same direction}}$$

Local diffusion equations

(1) $\partial_t u - \operatorname{div}(a(x, \nabla u)) + \beta(u) = 0$ in $\Omega \times (0,\infty)$

The linear model: $j(x,t) = -c \cdot \nabla u$

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- Fourier's law ~ u rep. heat density

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- Fourier's law ~ u rep. heat density
- Fick's law ~ u rep. chem. comp.

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The constant c = conductivity coefficient indicating how fast/slow heat/particles is/are transported.

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- $c(|\nabla u|) = |\nabla u|^{p-2}$, $1 \leq p < \infty$

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p-Laplace operator

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$\Delta_p u$: p-Laplace operator

▷ Ladyšenskaya (1967) — Movement of some non-Newtonian fluids

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p-Laplace operator

- Pélissier-Reynaud (1973) Smelting process of glaciers

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p-Laplace operator

- Perona-Malik (1990) — Image processing $1<p<2$

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p-Laplace operator

$p = 1: \quad \Delta_1 u = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ total variational flow operator

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The nonlinear model: $\quad j(x,t) = -c(u,\nabla u)\cdot\nabla u$

- $c(u) = m\,|u|^{m-1}, \quad m>0$

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Porous media operator

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Porous media operator

▷ Vazquez (Oxford Press book)

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1^{st} Main interest of this talk

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Regularisation effect:

$$u(0) \in L^q \;\Rightarrow\; u(t) \in L^\infty \text{ for all } \underline{\underline{t>0}}$$

$(q \geq 1)$

$$\& \quad \|u(t)\|_\infty \lesssim t^{-\delta_q} \|u(0)\|_q^{\gamma_q}$$

Why we are interested in L^1-L^∞ regul. effects?

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Why we are interested in L^1-L^∞ regul. effects?

- $u(t,\cdot) \in L^\infty \implies u(t,\cdot) \in C^\alpha$
- $u(t,\cdot) \in L^\infty \implies \frac{\partial u}{\partial t} \in L^1_{loc}$

Why we are interested in L^1-L^∞ regul. effects?

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- $u(t,\cdot) \in L^\infty \implies \frac{\partial u}{\partial t} \in L^1_{loc}$

In particular:

$$\boxed{\text{Entropy/mild solutions are strong.}}$$

Assumptions

Suppose $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying for $1 < p < \infty$:

- $|a(x,\xi)| \le \alpha(x)|\xi|^{p-1} + h(x)$
- $a(x,\xi)\,\xi \ge \eta|\xi|^p$
- $(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) > 0$ $\xi_1 \neq \xi_2$

for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Omega$, where $\eta > 0$, $\alpha \in L^\infty$ & $h \in L^{p'}$ with $p' = \frac{p}{p-1}$.

$u(0) \in L^q \Rightarrow u(t) \in L^\infty$ + uniform bdds

▷ $1 < p < d$:

$$\delta_q = \frac{\delta}{1 - \gamma\left(1 - \frac{q(d-p)}{d\, m_0}\right)}$$

for all $\frac{d(2-p)}{p} < q \le \frac{d m_0}{d-p}$

$m_0 \geq p$ s.t.

$$\gamma_q = \frac{\gamma\, q(d-p)}{d\, m_0\left(1 - \gamma\left(1 - \frac{q(d-p)}{d\, m_0}\right)\right)}$$

$$\left(\frac{d}{d-p} - 1\right) m_0 + p - 2 > 0$$

with $\delta = \frac{d-p}{p m_0 + (d-p)(p-2)}$ $\gamma = \frac{p m_0}{p m_0 + (d-p)(p-2)}$

(where, if $\frac{2d}{d+2} < p < d$ then $m_0 = p$ & if $\frac{2d}{d+1} < p < d$ then $1 \le q \le \frac{dp}{d-p}$)

$u(0) \in L^q \Rightarrow u(t) \in L^\infty$ + uniform bdds

$\triangleright$ $p = d$:

$$\delta_q = \frac{\delta_\theta}{1 - \gamma_\theta(1 - q/r_\theta)} \qquad \text{for every } 0 < \theta < 1$$

$$\& \quad 1 \le q \le \frac{2}{1-\theta}$$

$$\gamma_q = \frac{\gamma_\theta q}{r_\theta\left(1 - \gamma_\theta(1 - q/r_\theta)\right)}$$

with $r_\theta = \dfrac{2}{1-\theta}$, $\delta_\theta = \dfrac{1-\theta}{2\theta + p(1-\theta)}$, $\gamma_\theta = \dfrac{2}{2\theta + p(1-\theta)}$

$u(0) \in L^q \Rightarrow u(t) \in L^\infty$ + uniform bdds

▷ $p>d$:

$$\delta_q = \frac{\theta_0}{p(1-\beta(1-\frac{q}{2}))} \quad \text{for every}$$

$$1 \le q \le 2$$

$$\gamma_q = \frac{\beta q}{2-\beta(2-q)}$$

where $\theta_0 = \dfrac{pd}{pd+2(p-d)}$ & $\beta = \dfrac{2\theta_0 + p(1-\theta_0)}{p}$.

Where is the semigroup?

The operator $\mathcal{A}u := -\operatorname{div}(a(x,\nabla u))$ equipped with some boundary conditions

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The operator $\mathcal{A}u := -\mathrm{div}(a(x,\nabla u))$ equipped with some boundary conditions

& realised in L^q ($1 \le q < \infty$, usually $q=2$ or $q=1$) by

$$A = \{(u,v) \in L^q \times L^q \mid u \in D(\mathcal{A}) \;\&\; v = \mathcal{A}u\}$$

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& realised in L^q ($1 \le q < \infty$, usually $q=2$ or $q=1$) by

$$A = \{(u,v) \in L^q \times L^q \mid u \in D(\mathcal{A}) \ \&\ v = \mathcal{A}u\}$$

is either

m-completely accretive in L^q

or m-T-accretive in L^1 with complete resolvent (if $q=1$)

Where is the semigroup?

By the celebrated Crandall-Liggett theorem,

the abstract Cauchy problem

$$(\mathcal{CP}) \quad \begin{cases} \frac{du}{dt} + Au = 0 & \text{on } (0,\infty) \\ u(0) = u_0 \in \overline{D(A)}^{L^q} \end{cases}$$

is well-posed in the sense of mild solutions.

Where is the semigroup?

- A function $u \in C([0,\infty); L^q)$ is a mild solution of (EP) if for every $\varepsilon > 0$ and for every $T > 0$ and partition $0 = t_0 < t_1 < \cdots < t_N = T$ with $t_i - t_{i-1} < \varepsilon$ there is

$$u_\varepsilon(t) = u_0 \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} u_{\varepsilon,i} \mathbb{1}_{(t_{i-1}, t_i]}(t)$$

with coeff. $u_{\varepsilon,i}$ solving recursively $u_{\varepsilon,i} + (t_i - t_{i-1}) A u_{\varepsilon,i} \ni u_{\varepsilon,i-1}$ such that

$$\sup_{t \in [0,T]} \| u_\varepsilon(t) - u(t) \|_q \le \varepsilon.$$

Where is the semigroup?

For given $u_0 \in \overline{D(A)}$, let $u : [0,\infty) \to L^q$ be the the mild solution of (EP), then set

$$T_t u_0 := u(t) \quad \forall t \geq 0.$$

Where is the semigroup?

Then the family $\{T_t\}_{t\geq 0}$ of mappings $T_t : \overline{D(A)} \to \overline{D(A)}$ satisfies:

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Where is the semigroup?

Then the family $\{T_t\}_{t\geq 0}$ of mappings $T_t : \overline{D(A)} \to \overline{D(A)}$ satisfies:

- $T_t \circ T_s = T_{t+s}$ for every $t, s \geq 0$
- $\lim_{t\to 0+} \|T_t u - u\|_X = 0$ for every $u \in \overline{D(A)}^X$
- $\|T_t u - T_t \hat{u}\|_X \leq e^{\omega t} \|u - \hat{u}\|_X$ for all $u, \hat{u} \in \overline{D(A)}^X$

1^{st} Main interest of this talk

To investigate the "regularising effect":

For given $u \in L^q(\Sigma,\mu)$ for some $q \geq 1$,

$$T_t u \in L^r(\Sigma,\mu) \ \forall t>0 \ \& \text{ some } q<r\leq\infty$$

+ uniform bdds/decay rates as $t \to \infty$.

The story

The story

- Linear semigroup theory :

The story

▷ **Linear semigroup theory:**

Let $\{T_t\}_{t\geq 0}$ be a semigroup of linear operators $T_t \in \mathcal{L}(L^q(\Sigma,\mu))$ for $1\leq q\leq\infty$ of a σ-finite measure space (Σ,μ).

The story

- Nelson '66 : $\{T_t\}_{t\geq 0} \sim -H$, $H = -\Delta + x^2$ Hermite operator

E. NELSON, A quartic interaction in two dimensions, *in* "Mathematical Theory of Elementary Particles," pp. 69–73, MIT Press, Cambridge, Mass., 1966.

The story

- Nelson '66: $\{T_t\}_{t\geq 0} \sim -H$, $H = -\Delta + x^2$ Hermite operator

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$\{T_t\}_{t\geq 0}$ is said to be hypercontractive if for some (all) $1 < q < r < \infty$ there is $t_0 = t_0(q,r) > 0$ such that

$$T_{t_0} : L^q \to L^r.$$

The story

- B. Simon & Hoegh-Krohn '72:

B. Simon and R. Hoegh-Krohn, Hypercontractive semigroups and two-dimensional self-coupled Bose fields, *J. Funct. Anal.* **9** (1972), 121–180.

The story

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B. SIMON AND R. HOEGH-KROHN, Hypercontractive semigroups and two-dimensional self-coupled Bose fields, *J. Funct. Anal.* 9 (1972), 121–180.

Using Stein's & Riesz-Thorin interpolation techniques to obtain

$$T_{t_0} : L^{\tilde{q}} \to L^{\tilde{r}} \text{ for } \tilde{q} < q \text{ \& } \tilde{r} > r.$$

The story

- L. Gross '76:

L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.* **97** (1976), 1061–1083.

The story

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Characterisation of hypercontractivity in terms of logarithmic Sobolev inequalities

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Characterisation of hypercontractivity in terms of logarithmic Sobolev inequalities

$$\int_{\mathbb{R}^d} |f|^2 \ln |f| \, d\mu \le C \cdot \underbrace{\int_{\mathbb{R}^d} |\nabla f|^2 d\mu}_{= (-Af, f)_2} + \|f\|_2^2 \cdot \ln \|f\|_2$$

The story

- In the 80's: The focus shifted towards a a stronger property called ultracontractivity for all $t>0$, $T_t : L^1 \to L^\infty$.

The story

- E. B. Davies & B. Simon '84.

E. B. Davies and B. Simon, *Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. **59** (1984), 335–395. doi:10.1016/0022-1236(84)

The story

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$\{T_t\}_{t\geq 0}$ is ultra-contractive $\iff$ $-A \sim \{T_t\}$ satisfies one-parameter family of Log-Sobolev ineq.

The story

- Varopoulos '85:

N. T. Varopoulos, *Hardy-Littlewood theory for semigroups*, J. Funct. Anal. **63** (1985), 240–260. doi:10.1016/0022-1236(85)90087-4

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$\|T_t\|_{1\to\infty} \leq c \cdot t^{-\frac{d}{2}}$ for every $t>0$ $\iff$ $-A \sim \{T_t\}_{t\geq 0}$ satisfies a d-dim Sobolev inequality

The story

▷ **Nonlinear semigroup theory:**

Let $\{T_t\}_{t\geq 0}$ be a semigroup of mappings $T_t : L^q \to L^q$ for $1 \leq q \leq \infty$.

The story

- P. Bénilan '78

______, *Opérateurs accrétifs et semi-groupes dans les espaces* L^p $(1 \leq p \leq +\infty)$, Functional Analysis and Numerical Analysis, Japan-France seminar, H. Fujita (ed.), Japan Society for the Advancement of Science, 1978.

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$-A \sim \{T_t\}_{t\geq 0}$ satisfies a d-dim Sobolev inequality $\Longrightarrow$ $T_t : L^1 \to L^\infty$ for all $t > 0$

The story

- L. Véron '79

L. Véron, *Effets régularisants de semi-groupes non linéaires dans des espaces de Banach*, Ann. Fac. Sci. Toulouse Math. (5) **1** (1979), 171–200. Available at http://www.numdam.org/item?id=AFST_1979_5_1_2_171_0

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First, who obtained the L^1–L^∞ bounds

$$\|T_t u - T_t \hat{u}\|_\infty \le C\, t^{-\delta}\, \|u-\hat{u}\|_1^{\gamma}$$

& $$\|T_t u\|_\infty \le C\, t^{-\delta}\, \|u\|_1^{\gamma} \quad \text{for all } t>0.$$

from a one-parameter family of Sobolev-ineq.

for $\{T_t\}_{t\ge 0} \sim \Delta_p^D$ & $\{T_t\}_{t\ge 0} \sim \Delta^D(\cdot^m)$.

The story

- Cipriani & Grillo '01:

F. Cipriani and G. Grillo, *Uniform bounds for solutions to quasilinear parabolic equations*, J. Differential Equations **177** (2001), 209–234. doi:10.1006/jdeq.2000.3985

The story

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Adapted Gross' & Simon-Davies' approach to nonlinear diffusion equations.

The story

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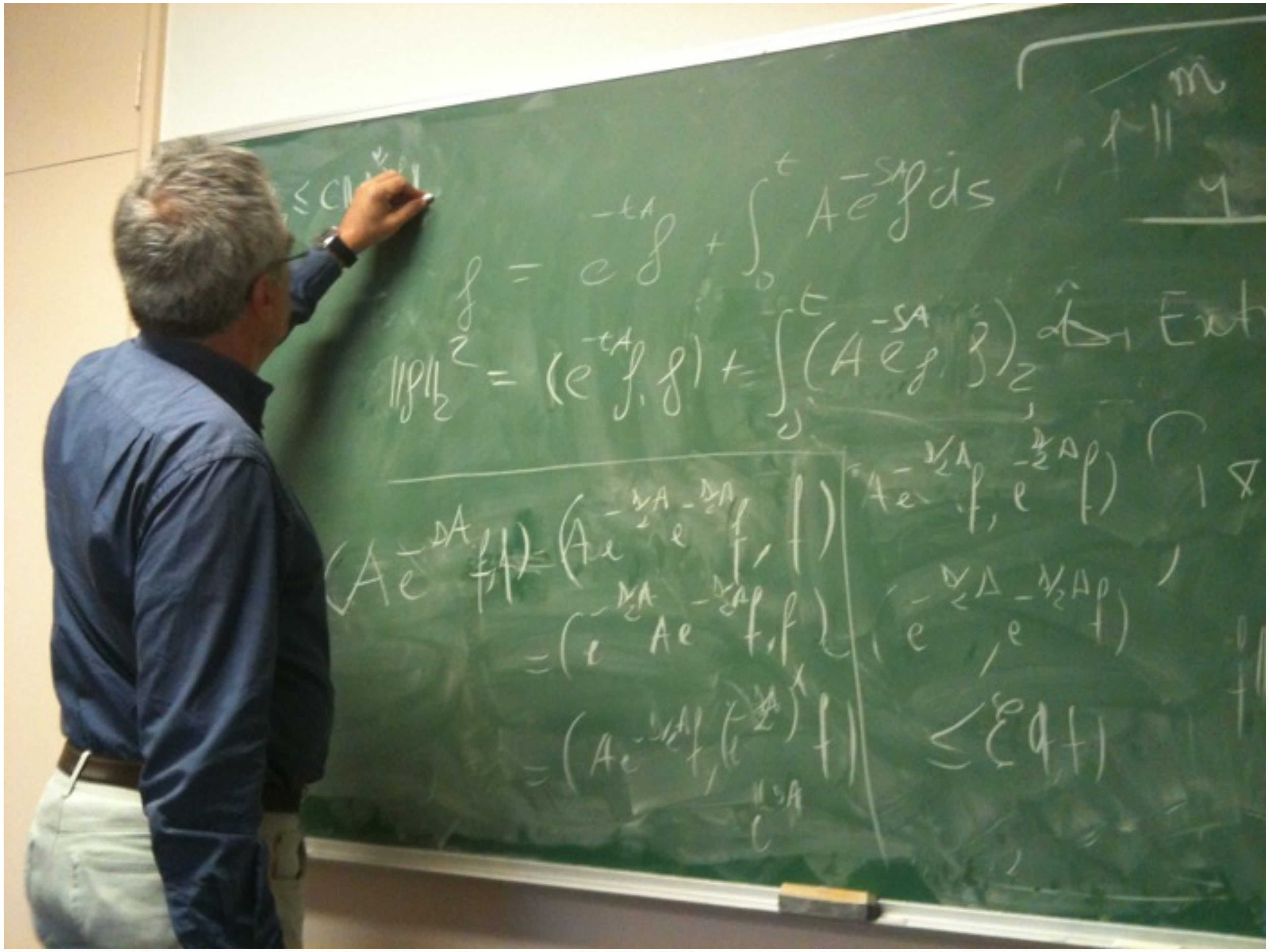
F. Cipriani and G. Grillo, *Uniform bounds for solutions to quasilinear parabolic equations*, J. Differential Equations **177** (2001), 209–234. doi:10.1006/jdeq.2000.3985

- ▷ Del Pino & Dolbeault '02, '04 + Gentil
- ▷ P. Takáč '04
- ▷ Bonforte & Grillo '05 ×2, '06
- ▷ Merkes '08, '09
- ▷ Warma '14



Log-Sobolev is not always the direct & simplest way

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2. When $r : [0,\infty) \to [p_0,\infty)$ is nondecreasing & C^1, then one shows by using the Log-Sobolev inequality that

$$s \mapsto y(s) := \log \|u(s)\|_{r(s)}$$

satisfies a differential inequality.

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$\Longrightarrow$ L^q-L^r-regularising effect of $u(0) \mapsto u(t)$, $t>0$.

The main idea to obtain L^{r_0}–L^{∞} estimates

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The main idea to obtain L^{r_0}–L^∞ estimates

- Construction of a one-parameter family of (Log-) Sobolev-inequalities:

$$\|u\|_{r_q}^{q} \leq C_q \cdot \langle Au, |u|^{q-p} u \rangle \quad \text{for all } q \geq p$$

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For $q \geq p$

$\langle -\Delta_p u , |u|^{q-p} u \rangle$

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$$p^* = \frac{pd}{d-p} \qquad \geq C_{qp} \left(\int_\Omega \left| |u|^{\frac{q-p}{p}} u \right|^{p^*} dx \right)^{\frac{p}{p^*}}$$

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$$\geq C_{qp} \left(\int_\Omega \left| |u|^{\frac{q-p}{p}} u \right|^{p^*} dx \right)^{\frac{p}{p^*}}$$

$\&$ $\kappa := \frac{d}{d-p} > 1$

$$= C_{qp} \left(\int_\Omega |u|^{\frac{qd}{d-p}} dx \right)^{\frac{d-p}{d}} = C_{qp} \|u\|_{q\kappa}^q$$

The main idea to obtain L^{r_0}–L^{∞} estimates

- Construction of a one-parameter family of (Log-) Sobolev-inequalities:

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The main idea to obtain L^{q_0}–L^∞ estimates

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- For every $q \geq p$, one deduces

$$\|T_t u\|_{K_q} \leq C_q^* \, t^{-\frac{1}{q}} \, \|u\|_{q-p+2}^{\frac{q-p+2}{q}} \quad \text{for all } t>0$$

The main idea to obtain L^{p_0}–L^{∞} estimates

- Construction of a one-parameter family of (Log-) Sobolev-inequalities:

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- For every $q \geq p$, one deduces

$$\|T_t u\|_{\kappa_q} \leq C_q^{*} \, t^{-\frac{1}{q}} \, \|u\|_{q-p+2}^{\frac{q-p+2}{q}} \quad \text{for all } t > 0$$

- Parabolic Moser iteration

Then, one constructs $(q_n)_{n\geq 0}$ by $\quad \kappa := \frac{d}{d-p} > 1$

$$q_0 \geq p$$

& $q_{n+1} = \kappa q_n + p - 1$ for every $n \in \mathbb{N}$.

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Then $q_{n+1} - q_n = \underbrace{\kappa^n}_{\kappa > 0} \Big[\underbrace{(\kappa - 1) q_0 + p - 2}_{> 0 \text{ by assumption}} \Big] > 0$ ∇

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$$\& \quad q_n = \frac{\kappa^n \left[(\kappa-1) q_0 + p - 2 \right]}{\kappa - 1} + \frac{2-p}{\kappa - 1} \nearrow +\infty$$

Inserting $q_{n+1} \longmapsto q$ into the one-parameter family of Sobolev inequalities yields

$$\|T_t u\|_{\kappa q_{n+1}} \le C_{q_{n+1}}^{*\,\frac{1}{q_{n+1}}}\; t^{-\frac{1}{q_{n+1}}}\, \|u\|_{\kappa q_n}^{\frac{\kappa q_n}{q_{n+1}}}$$

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For any sequence $(t_\nu)_{\nu\geq 0} \subseteq [0,1]$ s.th. $\sum_{\nu\geq 0} t_\nu = 1$ (e.g., $t_\nu = 2^{-(\nu+1)}$)

Inserting $q_{n+1} \longmapsto q$ into the one-parameter family of Sobolev inequalities yields

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$$\Longrightarrow \|T_{\sum_{\nu=0}^{n} t t_\nu} u\|_{\kappa q_{n+1}} \leq \prod_{\nu=0}^{n+1} C_{q_\nu}^{\frac{\kappa^{n+1-\nu}\, q_\nu}{q_{n+1}}} \times \prod_{\nu=0}^{n} t_\nu^{-\frac{\kappa^{n-\nu}}{q_{n+1}}} \times t^{-\frac{1}{q_{n+1}}\sum_{\nu=0}^{n}\kappa^\nu}\, \|u\|_{\kappa q_0}^{\frac{\kappa^{n+1} q_0}{q_{n+1}}}$$

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Letting $n \to \infty$

$$\|T_t u\|_\infty \lesssim t^{-\delta} \|u\|_{\kappa q_0}^{\sigma} \quad \text{for all } u \in L^{\kappa q_0}$$

with $\delta := \frac{1}{(\kappa-1)q_0 + p - 2}$, $\sigma = \frac{(\kappa-1)q_0}{(\kappa-1)q_0+p-2}$


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2nd Main interest of this talk

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What is known about the regularisation effect of non local diffussion equations?

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2^{nd} Main interest of this talk

1^{st} Example

$$\partial_t u - (-\Delta_p)^s u + \beta(u) = 0 \quad \text{in } \Omega \times (0,\infty)$$

where $$\langle -(-\Delta_p)^s u, v\rangle = \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x))}{|y-x|^{d+sp}}\,dx\,dy$$

for $1<p<\infty$ & $0<s<1$.

"fractional (weak) p-Laplace operator"

2^{nd} Main interest of this talk

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for $1<p<\infty$ & $0<s<1$. "fractional (weak) p-Laplace operator"

Under additional regularity assumptions on u &/or certain p & s, one has

$$(-\Delta_p)^s u(x) := \text{P.V.} \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{d+sp}}\,dy$$

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2^{nd} Example

$$
(1)\quad \begin{cases} \partial_t u - a(x,\nabla u)\cdot\vec{\nu} + \beta(u) = 0 & \text{on } \partial\Omega\times(0,\infty), \\ -\operatorname{div}(a(x,\nabla u)\nabla u) = 0 & \text{in } \Omega\times(0,\infty). \end{cases}
$$

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If $P : \partial\Omega \ni \varphi \longmapsto P\varphi := u$ unique solution of

$$(DP_\varphi)\quad \begin{cases} -\mathrm{div}(a(x,\nabla u)\nabla u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

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then $\Lambda\varphi := a(x,\nabla P\varphi)\cdot\vec{\nu}$ Dirichlet-to-Neumann operator

associated with $\mathcal{A}u := -\operatorname{div}(a(x,\nabla u))$.

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Note, (1) is equivalent to

$$(2) \qquad \partial_t \varphi + a(x,\nabla P\varphi)\cdot\vec{\nu} + \beta(\varphi) = 0 \quad \text{on } \partial\Omega\times(0,\infty)$$

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The construction of a one-parameter family of Sobolev inequalities

2^{nd} Main interest of this talk

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The construction of a one-parameter family of Sobolev inequalities

$$\langle A\varphi, |\varphi|^{q-p}\varphi\rangle = \int_\Omega |\nabla P\varphi|^{p-2}\nabla P\varphi \cdot \nabla P(|\varphi|^{q-p}\varphi)\, dx$$

BUT $\nabla P(|\varphi|^{q-p}\varphi) \neq |P\varphi|^{q-p} P\varphi$

2^{nd} Main interest of this talk

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The construction of a one-parameter family of Sobolev inequalities

does not work!

Theorem [Coulhon, H. '15] Sobolev $\Rightarrow L^q$-L^r-reg

Let $A+\omega I$ be m-completely accretive in $L^q(\Sigma,\mu)$ with dense domain & $0\in A0$ & $\omega\in\mathbb{R}_+$.

If there are $1\le q,r\le\infty$, $\sigma>0$ & $C>0$ s.th.

(Sob) $$\|u-\hat{u}\|_r^\sigma \le C\left([u-\hat{u},v-\hat{v}]_q+\omega\|u-\hat{u}\|_q^q\right)$$

for all $(u,v),(\hat{u},\hat{v})\in A$, then $\{T_t\}_{t\ge0}\sim -A$ satisfies

$$\|T_tu-T_t\hat{u}\|_r\le\left(\frac{C}{q}\right)^{\frac{1}{\sigma}}e^{\omega t(\frac{q}{\sigma}+1)}\,t^{-\frac{1}{\sigma}}\|u-\hat{u}\|_q^{\frac{q}{\sigma}}\quad\forall u,\hat{u}\in L^q$$

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Here, $[u,v]_q:=\int_\Sigma |u|^{q-2}u\cdot v\,d\mu$ if $q>1$.

Proof of Theorem 1:

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$$\overset{\text{Hyp.}}{\geq} \frac{q}{C} \int_0^t \|T_s u - T_s \hat{u}\|_r^\sigma \, ds$$

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$$
\begin{aligned}
\|u-\hat{u}\|_q^q &\geq \|u-\hat{u}\|_q^q - \|T_t u - T_t \hat{u}\|_q^q \\
&= -\int_0^t \frac{d}{ds}\|T_s u - T_s \hat{u}\|_q^q \, ds \\
&= (-q)\int_0^t \left\langle (u-\hat{u})_q , \frac{d}{ds}T_s u - \frac{d}{ds}T_s \hat{u} \right\rangle_{q',q} ds \\
&= q\int_0^t \left\langle (u-\hat{u})_q , A T_s u - A T_s \hat{u} \right\rangle_{q',q} ds \\
&\overset{\text{Hyp.}}{\geq} \frac{q}{C}\int_0^t \|T_s u - T_s \hat{u}\|_r^\sigma \, ds \\
&\geq \frac{q}{C}\, t \, \|T_t u - T_t \hat{u}\|_r^\sigma
\end{aligned}
$$

$\diamond$

*Theorem [Coulhon, H. '15] Sobolev $\Rightarrow L^q$–L^r-reg

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If there are $1\le q, r\le\infty$, $\sigma>0$ & $C>0$ s.th.

(Sob) $$\|u-u_0\|_r^\sigma \le C\left([u-u_0, v]_q + \omega\|u-u_0\|_q^q\right)$$

for all $(u,v)\in A$, then $\{T_t\}_{t\ge0} \sim -A$ satisfies

$$\|T_t u - u_0\|_r \le \left(\frac{C}{q}\right)^{\frac{1}{\sigma}} e^{\omega t(\frac{q}{\sigma}+1)}\, t^{-\frac{1}{\sigma}} \|u-u_0\|_q^{\frac{q}{\sigma}} \quad \forall u\in L^q$$

2 Theorem [Coulhon, H. '15] Sobolev ⇒ L^q-L^∞-reg.

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If there are $1\le q,r<\infty$, $\sigma>0$ & $C>0$ s.th. $\frac{r}{\sigma}>1$ &

$$\text{(Sob)} \qquad \|u-\hat{u}\|_r^\sigma \le C\left([u-\hat{u},v-\hat{v}]_q+\omega\|u-\hat{u}\|_q^q\right)$$

for all $(u,v),(\hat{u},\hat{v})\in A$, then $\{T_t\}_{t\ge0}\sim -A$ satisfies

$$\|T_t u-T_t\hat{u}\|_\infty \lesssim t^{-\delta}\, e^{\omega t(\beta^*+1)}\|u-\hat{u}\|_{\frac{r}{\sigma}m_0}^{\gamma} \quad \forall u,\hat{u}\in L^{\frac{r}{\sigma}m_0},$$

for any $m_0>\sigma$ satisfying $(\frac{r}{\sigma}-1)m_0+\sigma-q>0$

with

$$\delta=\frac{1}{(\frac{r}{\sigma}-1)m_0+\sigma-q},\quad \beta^*=\frac{(\frac{q}{\sigma}+1)\frac{r}{q}-1}{(\frac{r}{\sigma}-1)m_0+\sigma-q},\quad \gamma=\frac{(\frac{r}{\sigma}-1)m_0}{(\frac{r}{\sigma}-1)m_0+\sigma-q}.$$

Idea of the Proof of Theorem 2:

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One has

- $\|T_t u - T_t \hat{u}\|_r \leq \left(\frac{C}{q}\right)^{\frac{1}{\delta}} e^{\omega t\left(\frac{q}{\delta}+1\right)}\, t^{-\frac{1}{\delta}} \|u-\hat{u}\|_q^{\frac{q}{\delta}}$

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- $\|T_t u - T_t \hat{u}\|_\infty \leq e^{\omega t} \|u-\hat{u}\|_\infty \quad \forall\, u, \hat{u} \in L^q \cap L^\infty$

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A new "nonlinear interpolation theorem"

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A new "nonlinear interpolation theorem"

$\Longrightarrow$ for every $0<\theta<1$ there is $\eta(\theta) \in (0,1)$ s.th.

$$\|T_t u - T_t \hat{u}\|_{\frac{r}{1-\theta}} \leq \left[\frac{q/\sigma}{(1-\theta)\, q/\sigma + \theta}\right]^{\frac{1-\theta}{r}} \times \left[C \cdot e^{\omega t \tilde{\beta}}\, t^{-\sigma}\right]^{1-\theta} \times$$

$$\times\, C_{\infty,q}^{*\,(1-\theta)\frac{q}{\sigma}+\theta}$$

$$\times \|u-\hat{u}\|_{\frac{q}{1-\eta(\theta)}}^{(1-\theta)\frac{q}{\sigma}+\theta}$$

Idea of the Proof of Theorem 2:

With $\kappa = \frac{r}{b} > 1$

take $\Theta_m = 1 - \frac{1}{m} \cdot \frac{r}{\kappa}$ $\forall\, m \geq m_0$

s.t. $\frac{q}{1 - z(\Theta_m)} \geq 1$

$\square$

2* Theorem [Coulhon, H. '15] Sobolev ⇒ L^q–L^∞-reg.

Let $A+\omega I$ be m-completely accretive in $L^q(\Sigma,\mu)$ with dense domain & $0\in A u_0$ & $\omega\in\mathbb{R}_+$.

If there are $1\le q, r<\infty$, $\sigma>0$ & $c>0$ s.th. $\frac{r}{\sigma}>1$ &

$$\text{(Sob)}\qquad \|u-u_0\|_r^\sigma \le C\left([u-u_0, v]_q + \omega\|u-u_0\|_q^q\right)$$

for all $(u,v)\in A$, then $\{T_t\}_{t\ge 0}\sim -A$ satisfies

$$\|T_t u-u_0\|_\infty \lesssim t^{-\delta} e^{\omega t(\beta^*+1)}\|u-u_0\|^{\gamma}_{\frac{r}{\sigma}m_0}\quad\forall\, u\in L^{\frac{r}{\sigma}m_0},$$

for any $m_0>\sigma$ satisfying $\left(\frac{r}{\sigma}-1\right)m_0+\sigma-q>0$

with

$$\delta=\frac{1}{\left(\frac{r}{\sigma}-1\right)m_0+\sigma-q},\quad \beta^*=\frac{\left(\frac{q}{\sigma}+1\right)\frac{r}{q}-1}{\left(\frac{r}{\sigma}-1\right)m_0+\sigma-q},\quad \gamma=\frac{\left(\frac{r}{\sigma}-1\right)m_0}{\left(\frac{r}{\sigma}-1\right)m_0+\sigma-q}.$$

3 Theorem [Coulhon, H. '15] Extrapolation towards L^1

For $1 \le q < r \le \infty$, let $\{T_t\}_{t\ge 0}$ be L^1-contractive semigroup on $L^1 \cap L^r(\Sigma,\mu)$, $T_t 0 \equiv 0$ & suppose, there are $\delta, \gamma > 0$ s.th.

$$\|T_t u - T_t \hat{u}\|_r \le C\, t^{-\delta} \|u-\hat{u}\|_q^{\gamma} \quad \forall t>0 \ \& \ \forall u, \hat{u} \in L^1 \cap L^r$$

For $\Theta_r := \frac{r-q}{q(r-1)} > 0$ if $r<\infty$ & $\Theta_\infty := \frac{1}{q}$ if $r=\infty$

assume that $\gamma(1-\Theta_r) < 1$.

Then
$$\|T_t u - T_t \hat{u}\|_r \le (2^\delta C)^{1/\Theta}\, t^{-\frac{\delta}{\Theta}} \|u-\hat{u}\|_1^{\gamma\frac{\Theta_r}{\Theta}} \quad \forall u, \hat{u} \in L^1(\Sigma,\mu)$$

with $\Theta := 1-\gamma(1-\Theta_r)$.

THE UNIVERSITY OF SYDNEY

Daniel Hauer

Thank You!

Applications

Applications

1. Total variational flow in $\mathbb{R}^d$

$$A := \left\{ (u,v) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \;\middle|\; \begin{array}{l} u \in BV(\mathbb{R}^N) \;\&\; \exists z \in L^\infty(\mathbb{R}^d, \mathbb{R}^d),\, \|z\|_\infty \leq 1, \\ v = -\operatorname{div} z \text{ in } \mathcal{D}' \;\&\; (z, Du) = |Du| \end{array} \right\}$$

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Sobolev in

$$\|u\|_{L^{\frac{d}{d-1}}} \le C \, |Du|(\mathbb{R}^d)$$

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1. Total variational flow in $\mathbb{R}^d$

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Thm 1* $\Longrightarrow$ $$\|T_t u\|_{\frac{d}{d-1}} \leqslant \frac{c}{2}\, t^{-1} \|u\|_2^2 \quad \forall t > 0 \;\; \& \; \forall u \in L^2(\mathbb{R}^d)$$

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$$\overset{\text{Thm } 1^*}{\Longrightarrow} \quad \|T_t u\|_{\frac{d}{d-1}} \leqslant \frac{c}{2}\, t^{-1} \|u\|_2^2 \quad \forall t>0$$
$$\& \ \forall u \in L^2(\mathbb{R}^d)$$

Since $d/d-1 > 1$,

$$\overset{\text{Thm } 2^*}{\Longrightarrow} \quad \|T_t u\|_\infty \lesssim t^{-\frac{1}{(d/(d-1)-1)m_0-1}} \|u\|_{\frac{dm_0}{d-1}}^{\frac{(\frac{d}{d-1}-1)m_0}{(d/(d-1)-1)m_0-1}} \quad \forall u \in L^{\frac{dm_0}{d-1}}$$

for any $m_0 > 1$ s.th. $\left(\frac{d}{d-1}-1\right)m_0 - 1 > 0$.

Where is the semigroup?

The operator $Au := -\mathrm{div}(a(x,\nabla u))$ equipped with some boundary conditions

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty) \qquad \text{(homog. Dirichlet)}$$

$$a(x,\nabla u)\cdot \vec{\nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty) \qquad \text{(homog. Neumann)}$$

$$a(x,\nabla u)\cdot\vec{\nu} + b(x,u) = 0 \quad \text{on } \partial\Omega\times(0,\infty) \qquad \text{(homog. Robin)}$$

$$\lim_{|x|\to\infty} u(x,t) = 0 \quad \text{for } t>0 \qquad \text{(vanishing at } \infty)$$

Where is the semigroup?

- An operator $A \subseteq L^q \times L^q$ is called accretive if for every $\lambda > 0$, the resolvent $J_\lambda := (1+\lambda A)^{-1}$ satisfies

$$\|J_\lambda u - J_\lambda \hat{u}\|_q \le \|u - \hat{u}\|_q \quad \text{for all } u, \hat{u} \in Rg(1+\lambda A)$$

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- An operator $A \subseteq L^1 \times L^1$ is m-T-accretive with complete resolvent
  if A is m-accretive in L^1, J_λ is order-preserving & $\|J_\lambda u\|_q \leq \|u\|_q$ for all $1 \leq q \leq \infty$ & $\lambda > 0$.