Conjugacy Classes in Finite Conformal Orthogonal Groups

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This treatment of the conformal orthogonal groups is modelled on the report 'Conjugacy Classes in Finite Orthogonal Groups' combined with the report 'Conjugacy Classes in Finite Conformal Symplectic Groups'. It is a revision the package code of Scott Murray in `orthogonal.m`

The conjugacy classes are obtained by first computing a complete collection of invariants and then determining a representative matrix for each invariant.

A partial analysis of similar algorithms for unitary groups can be found in [2]. There are some remarks about the conformal orthogonal groups in the unpublished draft [4]. A more extended account is in Chapter 5 of Britnell's thesis [1] and a description of the invariants, based on the work of Wall [7], Springer and Steinberg [6] and Milnor [3] can be found in the Shinoda's paper [5].

1 Conformal orthogonal groups

In MAGMA a conformal orthogonal group in dimension n over the field $k = \mathrm{GF}(q)$, where q is odd, consists of $n \times n$ matrices A over k such that for each A there is a non-zero element $\phi = \phi(A)$ in k such that $AJA^{\mathrm{tr}} = \phi(A)J$, where A^{tr} denotes the transpose of A and J is a 'standard' non-degenerate symmetric matrix. For all n there are two isometry classes of symmetric bilinear forms. However if n is odd, up to isomorphism, there is just one orthogonal group, whereas if n is even there are two orthogonal groups.

If $V = k^n$, the matrix J defines a symmetric bilinear form on V given by $\beta(u, v) = uJv^{\mathrm{tr}}$.

It is immediate that $\phi : \mathrm{CGO}(V) \to k^\times$ is a homomorphism. If Q is the subgroup $\left\{ \begin{pmatrix} aI & 0 \\ 0 & I \end{pmatrix} \middle| \; a \in k^\times \right\}$, then $Q \cap \mathrm{GO}(V) = 1$ and $\mathrm{CGO}(V) = \mathrm{GO}(V)Q$. It follows that ϕ is surjective and its kernel is $\mathrm{GO}(2n, q)$. That is, $g_1, g_2 \in \mathrm{CGO}(V)$ are in the same coset of $\mathrm{GO}(V)$ if and only if $\phi(g_1) = \phi(g_2)$. The centre Z of $\mathrm{CGO}(V)$ is the group of $q - 1$ non-zero scalar matrices and $|\mathrm{CGO}(V) : Z \circ \mathrm{GO}(V)| = 2$. Let $\mathrm{CGO}_\phi(V)$ denote the coset of elements of $\mathrm{CGO}(V)$ with multiplier ϕ.

The description of the conjugacy classes in $\mathrm{CGO}(V)$ closely parallels the descriptions of the conjugacy classes in the general linear groups, the symplectic groups and the orthogonal groups defined on V.

References

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