

# ON THE CARTWRIGHT-STEGER SURFACE

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**Abstract** *In this article, we study various concrete algebraic and differential geometric properties of the Cartwright-Steger surface, the unique smooth surface of Euler number 3 which is neither a projective plane nor a fake projective plane. In particular, we determine the genus of a generic fiber of the Albanese fibration, and deduce that the singular fibers are not totally geodesic, answering an open problem about fibrations of a complex ball quotient over a Riemann surface.*

## 0. INTRODUCTION

The Cartwright-Steger surface was found during work on the classification of fake projective planes completed in [PY] and [CS1]. A fake projective plane is a smooth surface with the same Betti numbers as the projective plane but not biholomorphic to it. It is known that a fake projective plane is a complex two ball quotient  $\Pi \backslash B_{\mathbb{C}}^2$  with Euler number 3, where  $\Pi$  is an arithmetic lattice in  $\mathrm{PU}(2, 1)$ , cf. [PY]. In the scheme of classification of fake projective planes carried out in [PY], the lattices  $\Pi$  are torsion-free subgroups having finite abelianization and of a certain index  $N$  in certain maximal arithmetic subgroups  $\bar{\Gamma}$  of  $\mathrm{PU}(2, 1)$  associated with a pair of number fields coming from a short finite list. The calculations reported in [CS1] found all torsion-free subgroups  $\Pi$  of index  $N$  (up to conjugacy) in each of these situations. All had finite abelianizations, and so gave a fake projective plane, with one exception, a subgroup  $\Pi$  of index 864 in a group  $\bar{\Gamma}$  associated with the pair  $\mathcal{C}_{11} = (\mathbb{Q}(\sqrt{3}), \mathbb{Q}(e^{\pi i/6}))$  on that list. The smooth projective surface  $X = \Pi \backslash B_{\mathbb{C}}^2$  is the subject of study in this article.

From an algebraic geometric point of view, the fake projective planes and the Cartwright-Steger surfaces are interesting since they have the smallest possible Euler number, namely 3, among smooth surfaces of general type, and constitute all such surfaces (cf. §2.1 of [Y2]). From a differential geometric point of view, they are interesting since they constitute smooth complex hyperbolic space forms, or complex ball quotients, of smallest volume in complex dimension two. We refer the reader to [R], [Y1], and [Y2] for some general discussions related to the above facts. Unlike fake projective planes, whose lattices arise from division algebras of non-trivial degree as classified, the Cartwright-Steger surface is defined by Hermitian forms over the number fields mentioned above. It is realized among experts that such a surface is commensurable to a Deligne-Mostow surface, the type of surfaces which have been studied by Picard, Le Vavas seur, Mostow, Deligne-Mostow, Terada and many others, cf. [DM1].

Even though the lattice involved is described in [CS2], it is surprising that the algebraic geometric structures of the surface are far from being understood. A typical problem is to find out the genus of a generic fiber of the associated Albanese fibration. Conventional algebraic geometric techniques do not seem to be readily applicable to such a problem. The goal of this article is to develop tools and techniques which allow us to understand concrete surfaces such as the Cartwright-Steger surface. In particular, we recover algebraic geometric

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properties from a description of the fundamental group of the surface, using a combination of various algebraic geometric, differential geometric, group theoretical techniques and computer implementations.

Here are the results obtained in this paper.

**Main Theorem** *Let  $X$  be the Cartwright-Steger surface and  $\alpha : X \rightarrow T$  the Albanese map.*

- (a) *The numerical invariants of  $X$  are*

$$c_1^2 = 9, \quad c_2 = 3, \quad \chi(\mathcal{O}_X) = 1, \quad q = h^{1,0} = 1, \quad p_g = h^{2,0} = 1, \quad h^{1,1} = 3.$$

*Moreover,  $H^2(X, \mathbb{Z})$  is torsion free.*

- (b) *The genus of a generic fiber of  $\alpha$  is 19.*  
(c) *All fibers of  $\alpha$  are reduced.*  
(d) *The Albanese torus  $T$  is  $\mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ , where  $\omega$  is a cube root of unity.*  
(e) *The Picard number of  $X$  is 3, equal to  $h^{1,1}(X)$ , so that all the Hodge  $(1,1)$  classes are algebraic. The Néron-Severi group is generated over  $\mathbb{Q}$  by three immersed totally geodesic curves we explicitly give.*  
(f) *The automorphism group  $\Sigma$  of  $X$ , isomorphic to  $\mathbb{Z}_3$ , has 9 fixed points, and induces a nontrivial action on  $T$  which has 3 fixed points. Three fixed points of  $\Sigma$  lie over each fixed point in  $T$ . Over one fixed point on  $T$ , the three fixed points of  $\Sigma$  are of type  $\frac{1}{3}(1,1)$ . The other 6 fixed points of  $\Sigma$  are of type  $\frac{1}{3}(1,2)$ .*

The Main Theorem follows from Lemma 3 for (a), Theorem 3 for (b), Corollary 2 for (c), Lemma 11 for (d), Corollary 1 and Lemma 8 for (e), and Proposition 11 for (f). As an immediate consequence, see Theorem 4, we have answered an open problem communicated to us by Ngaiming Mok on properties of fibrations on complex ball quotients.

**Corollary** *There exists a fibration of a smooth complex two ball quotient over a smooth Riemann surface with non-totally geodesic singular fibers.*

Here are a few words about the presentation of the article. To streamline our arguments and to make the results more understandable, we state and prove the geometric results of the article sequentially in the sections 3 to 5 of the article. Many of the results rely on computations in the groups  $\Pi$  and  $\bar{\Gamma}$ , often obtained with assistance of the algebra package Magma, and we present these exclusively in the first two sections of the paper (except for the proof of Proposition 11) with a geometric perspective each time it is possible. More details appear in a longer version of this paper and on the webpage of the first author [CKY].

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Since completing this paper, we were informed by Domingo Toledo that he, Fabrizio Catanese, JongHae Keum and Matthew Stover had independently proved some of our results in a paper they are preparing.

## 1. BASIC FACTS

**1.1.** Let  $F$  be a Hermitian form on  $\mathbb{C}^3$  with signature  $(2,1)$ . We denote by  $U(F) = \{g \in GL(3, \mathbb{C}) \mid g^* F g = F\}$  the subgroup of  $GL(3, \mathbb{C})$  preserving the form  $F$ , by  $SU(F)$  the subgroup of  $U(F)$  of elements with determinant 1, and by  $PU(F)$  their image in  $PGL(3, \mathbb{C})$ .

The group  $\mathrm{PU}(F)$  is naturally identified with the group of biholomorphisms of the two-ball  $B_{\mathbb{C}}^2(F) := \{[z] \in \mathbb{P}_{\mathbb{C}}^2 = \mathbb{P}(\mathbb{C}^3) \mid F(z) < 0\}$ .

Our aim is to study a special complex hyperbolic surface  $X = \Pi \backslash B_{\mathbb{C}}^2(F)$  where  $\Pi$  is a cocompact torsion-free lattice in some  $\mathrm{PU}(F)$ . The group  $\Pi$  appears as a finite index subgroup of an arithmetic lattice  $\bar{\Gamma}$  which can be easily described as follows.

Let  $\zeta = \zeta_{12}$  be a primitive 12-th root of unity. Then  $r = \zeta + \zeta^{-1}$  is a square root of 3. Let  $\ell = \mathbb{Q}(\zeta)$  and  $k = \mathbb{Q}(r) \subset \ell$ . For real and complex calculations below, we take  $\zeta = e^{\pi i/6}$ , and then  $r$  is the positive square root of 3. We could define  $\bar{\Gamma}$  to be the group of  $3 \times 3$  matrices  $g'$  with entries in  $\mathbb{Z}[\zeta]$  such that  $g'^* F' g' = F'$ , where

$$F' = \begin{pmatrix} r+1 & -1 & 0 \\ -1 & r-1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and  $g'^*$  is the conjugate transpose of  $g'$ , modulo  $Z = \{\zeta^j I : j = 0, \dots, 11\}$ .

However, it is convenient to work with a diagonal form instead of  $F'$ . Notice that  $F' = (r-1)^{-1} \gamma_0^* F \gamma_0$  for

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-r \end{pmatrix}, \quad \text{and} \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1-r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So we instead define  $\bar{\Gamma}$  to be the group of matrices  $g$ , modulo  $Z$ , with entries in  $\ell$ , which are unitary with respect to  $F$  for which  $g' = \gamma_0^{-1} g \gamma_0$  has entries in  $\mathbb{Z}[\zeta]$ . Such  $g$ 's have entries in  $\frac{1}{r-1} \mathbb{Z}[\zeta] \subset \frac{1}{2} \mathbb{Z}[\zeta]$ .

Since  $F$  is diagonal, it is easy to make the group  $\mathrm{PU}(F)$  act on the standard unit two-ball, which we will just denote by  $B_{\mathbb{C}}^2$ : if  $gZ \in \bar{\Gamma}$ , the action of  $gZ$  on  $B_{\mathbb{C}}^2$  is given by

$$(gZ).(z, w) = (z', w') \quad \text{if} \quad DgD^{-1} \begin{pmatrix} z \\ w \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} z' \\ w' \\ 1 \end{pmatrix},$$

for some  $\lambda \in \mathbb{C}$ , where  $D$  is the diagonal matrix with diagonal entries 1, 1 and  $\sqrt{r-1}$ . We will ignore the distinction between matrices  $g$  and elements  $gZ$  of  $\bar{\Gamma}$ .

Now  $\bar{\Gamma}$  contains a subgroup  $K$  of order 288 generated by the two matrices  $u = \gamma_0 u' \gamma_0^{-1}$  and  $v = \gamma_0 v' \gamma_0^{-1}$  where

$$u' = \begin{pmatrix} \zeta^3 + \zeta^2 - \zeta & 1 - \zeta & 0 \\ \zeta^3 + \zeta^2 - 1 & \zeta - \zeta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v' = \begin{pmatrix} \zeta^3 & 0 & 0 \\ \zeta^3 + \zeta^2 - \zeta - 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A presentation for  $K$  is given by the relations  $u^3 = v^4 = 1$  and  $(uv)^2 = (vu)^2$ . The elements of  $K$  are most neatly expressed if we use not only the generators  $u$  and  $v$ , but also  $j = (uv)^2$ , which is the diagonal matrix with diagonal entries  $\zeta$ ,  $\zeta$  and 1, and generates the center of  $K$ .

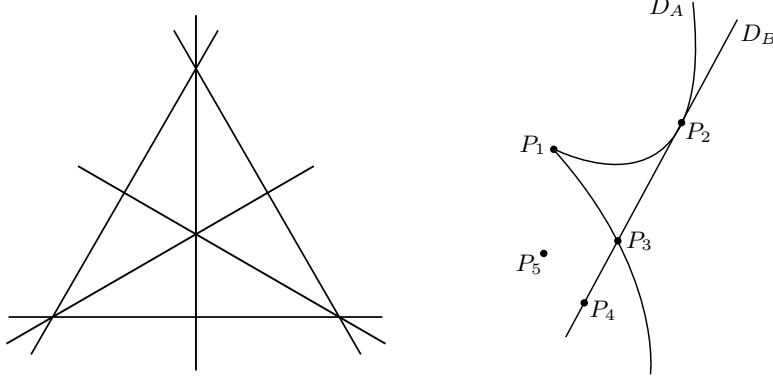
There is one further generator needed for  $\bar{\Gamma}$ , namely  $b = \gamma_0 b' \gamma_0^{-1}$  for

$$b' = \begin{pmatrix} 1 & 0 & 0 \\ -2\zeta^3 - \zeta^2 + 2\zeta + 2 & \zeta^3 + \zeta^2 - \zeta - 1 & -\zeta^3 - \zeta^2 \\ \zeta^2 + \zeta & -\zeta^3 - 1 & -\zeta^3 + \zeta + 1 \end{pmatrix}.$$

The matrices  $u'$ ,  $v'$  and  $b'$  have entries in  $\mathbb{Z}[\zeta]$  and are unitary with respect to  $F'$ . We shall always work below with their respective conjugates  $u$ ,  $v$  and  $b$  (unitary with respect to  $F$ ).

**Theorem 1** ([CS2]). *A presentation of  $\bar{\Gamma}$  is given by the generators  $u$ ,  $v$  and  $b$  and the relations*

$$u^3 = v^4 = b^3 = 1, \quad (uv)^2 = (vu)^2, \quad vb = bv, \quad (buv)^3 = (buvu)^2 v = 1.$$

FIGURE 1.  $\bar{Q} = \mathbb{P}_{\mathbb{C}}^2$  and  $R = \mathbb{P}_{\mathbb{C}}^2/\Sigma_3$ 

**1.2.** As communicated to us by John Parker,  $\bar{\Gamma}$  is isomorphic to a group generated by complex reflections first discovered by Mostow, denoted by  $\Gamma_{3, \frac{1}{3}}$  in [Mo1] and by  $\Gamma_{3,4}$  in [Pa]. See [CS2] for an explicit isomorphism.

It is also convenient to see  $\bar{\Gamma}$  as a (Deligne-)Mostow group: it corresponds to item 8 in the paper of Mostow [Mo2, p. 102] whose associated weights  $(2, 2, 2, 7, 11)/12$  satisfy the condition  $(\Sigma\text{INT})$  in the notation of [Mo2]. We refer to [Mo2] and [DM2] for details on the description below.

The orbifold quotient  $R := \bar{\Gamma} \backslash B_{\mathbb{C}}^2$  is a compactification of the moduli space of 5-tuples of distinct points  $(x_0, x_1, x_2, x_3, x_4) \in (\mathbb{P}_{\mathbb{C}}^1)^5$  modulo the diagonal action of  $\text{PGL}(2, \mathbb{C})$  and the action of the symmetric group on three letters  $\Sigma_3$  on the three first points. The compactification can be described as follows. First, it can be easily seen that the moduli space  $Q$  of 5-tuples of distinct points  $(x_0, x_1, x_2, x_3, x_4) \in (\mathbb{P}_{\mathbb{C}}^1)^5$  modulo the diagonal action of  $\text{PGL}(2, \mathbb{C})$  can be realized as  $\mathbb{P}_{\mathbb{C}}^2$  with a configuration of six lines removed. In homogeneous coordinates  $[X_0 : X_1 : X_2]$  on  $\mathbb{P}_{\mathbb{C}}^2$ , these six lines correspond to the three lines of “type A” with equation  $X_i = X_j$  ( $1 \leq i < j \leq 2$ ) and the three lines of “type B” with equation  $X_i = 0$  ( $i = 0, 1, 2$ ). In fact, the compactification  $\bar{Q} = \mathbb{P}_{\mathbb{C}}^2$  of  $Q$  is determined by the fact that we allow two or three of the points  $x_0, x_1$  and  $x_2$  to coincide ( $x_0 = x_1$  corresponds to  $X_0 = X_1$ ,  $x_0 = x_2$  to  $X_0 = X_2$  and  $x_1 = x_2$  to  $X_1 = X_2$ ) and we also allow one or two of the points  $x_0, x_1$  and  $x_2$  to coincide with  $x_3$  ( $x_0 = x_3$  corresponds to  $X_0 = 0$ ,  $x_1 = x_3$  to  $X_1 = 0$  and  $x_2 = x_3$  to  $X_2 = 0$ ).

Then, as we mentioned above, the underlying topological space of  $\bar{\Gamma} \backslash B_{\mathbb{C}}^2$  is a compactification  $R$  of  $Q/\Sigma_3$  and actually is the weighted projective plane  $\mathbb{P}(1, 2, 3) \cong \mathbb{P}_{\mathbb{C}}^2/\Sigma_3$  where the symmetric group on three letters  $\Sigma_3$  acts by permutation of the homogeneous coordinates  $[X_0 : X_1 : X_2]$  on  $\mathbb{P}_{\mathbb{C}}^2$ . There are two remarkable (irreducible) divisors on  $\mathbb{P}(1, 2, 3)$ : one is the image  $D_A$  of the divisors of type A, the other one is the image  $D_B$  of the divisors of type B. The divisor  $D_A$  has a cusp at the image  $P_1$  of the point  $[1 : 1 : 1]$  and the divisor  $D_B$  is smooth. These two divisors meet at two points: once at the image  $P_2$  of the points  $[1 : 0 : 0]$ ,  $[0 : 1 : 0]$  or  $[0 : 0 : 1]$  where they are tangent, once at the image  $P_3$  of the points  $[1 : 1 : 0]$ ,  $[1 : 0 : 1]$  or  $[0 : 1 : 1]$  where the intersection is transverse. There are also two singular points on  $\mathbb{P}(1, 2, 3)$ : one is a singularity of type  $A_1$  and is the image  $P_4 \in D_B$  of the points  $[1 : -1 : 0]$ ,  $[1 : 0 : -1]$  or  $[0 : 1 : -1]$ , the other one is a singularity of type  $A_2$  and is the image  $P_5$  of the points  $[1 : \omega : \omega^2]$  or  $[1 : \omega^2 : \omega]$  where  $\omega$  is a primitive 3rd root of unity.

**Remark 1.** In the book [DM2, p. 111], the divisor  $D_A$  (resp.  $D_B$ ) is denoted by  $D_{AA}$  (resp.  $D_{AB}$ ) and the points  $P_1, \dots, P_5$  simply by  $1, \dots, 5$ .

There is a standard method to compute the weight of the orbifold divisors on  $\bar{\Gamma} \backslash B_{\mathbb{C}}^2$  as well as the local groups at the orbifold points, according to the weights  $(2, 2, 2, 7, 11)/12$

mentioned above, in the notation of [Mo2]. The weight of  $D_A$  is  $3 = 2(1 - (2 + 2)/12)^{-1}$  and the weight of  $D_B$  is  $4 = (1 - (2 + 7)/12)^{-1}$ . This means that the preimage of  $D_A$  (resp.  $D_B$ ) in  $B_{\mathbb{C}}^2$  is a union of mirrors of complex reflections of order 3 (resp. 4). We will denote by  $\mathcal{M}_A$  (resp.  $\mathcal{M}_B$ ) the corresponding sets of mirrors and we will refer to these sets as *mirrors* of types  $A$  (resp.  $B$ ). Said another way, the isotropy group at a generic point of some  $M \in \mathcal{M}_A$  is isomorphic to  $\mathbb{Z}_3$  and the isotropy group at a generic point of some  $M \in \mathcal{M}_B$  is isomorphic to  $\mathbb{Z}_4$ , both generated by a complex reflection of the right order. This has to be compared with the description of  $\bar{\Gamma}$  as  $\Gamma_{3,4}$ .

The isotropy group at a point above the transverse intersection  $P_3$  of  $D_A$  and  $D_B$  is naturally isomorphic to  $\mathbb{Z}_3 \times \mathbb{Z}_4$ . As  $P_5$  is a singularity of type  $A_2$  but does not belong to any orbifold divisor, the local group at  $P_5$  is isomorphic to  $\mathbb{Z}_3$ . But since  $P_4 \in D_A$  is a singularity of type  $A_1$ , the local group at  $P_4$  has order  $8 = 2 \cdot 4$  and actually is isomorphic to  $\mathbb{Z}_8$  (see the discussion before Lemma 2 below).

It is a little bit more difficult to determine the isotropy group above the points  $P_1$  and  $P_2$ . It will also be useful to describe the stabilizer in  $\bar{\Gamma}$  of a mirror. For this, one can use a method similar to the one in [Der1, Lemma 2.12] and obtain the following lemma which already appeared in an unpublished manuscript of Deraux and Yeung.

**Lemma 1.** *Let  $\mathcal{M}_A$  (resp.  $\mathcal{M}_B$ ) denote the set of mirrors of complex reflections of order 3 (resp. 4) in  $\bar{\Gamma}$ .*

*Let  $\mathcal{P} \subset B_{\mathbb{C}}^2$  denote the set of points above  $P_1$  and  $\mathcal{T} \subset B_{\mathbb{C}}^2$  denote the set of points above  $P_2$ . The following holds.*

- (a) *The group  $\bar{\Gamma}$  acts transitively on  $\mathcal{M}_A$ , on  $\mathcal{M}_B$ , on  $\mathcal{P}$  and on  $\mathcal{T}$ .*
- (b) *For each point  $\xi \in \mathcal{P}$ , the stabilizer of  $\xi$  is the one labelled #4 in the Shephard-Todd list. It is a central extension of a  $(2, 3, 3)$ -triangle group, with center of order 2, and has order 24. There are precisely 4 mirrors in  $\mathcal{M}_A$  through each such  $\xi \in \mathcal{P}$ .*
- (c) *For each point  $\xi \in \mathcal{T}$ , the stabilizer of  $\xi$  is the one labelled #10 in the Shephard-Todd list. It is a central extension of a  $(2, 3, 4)$ -triangle group, with center of order 12, and has order 288. Through each such  $\xi \in \mathcal{T}$ , there are 8 elements of  $\mathcal{M}_A$  and 6 elements of  $\mathcal{M}_B$ .*
- (d) *The stabilizer of any element  $M \in \mathcal{M}_A$  is a central extension of a  $(2, 4, 12)$ -triangle group, with center of order 3.*
- (e) *The stabilizer of any element  $M \in \mathcal{M}_B$  is a central extension of a  $(2, 3, 12)$ -triangle group, with center of order 4.*

*Sketch of proof.* (a) Follows from the above discussion.

(b) The point  $P_1$  corresponds to  $x_0 = x_1 = x_2$  so that the computation  $3/2 = (1 - (2 + 2)/12)^{-1}$  shows that the spherical triangle group associated to the projective action of the isotropy group at  $\xi \in \mathcal{P}$  is  $(2, 3, 3)$ . Indeed, we have to consider the triangle with angles  $(2\pi/3, 2\pi/3, 2\pi/3)$  and take the symmetry into account (i.e. dividing the triangle into six parts), so that we obtain a triangle with angles  $(\pi/2, \pi/3, \pi/3)$ . The center has order given by  $2 = (1 - (2 + 2 + 2)/12)^{-1}$ . Comparing with [ST, Table 1], we see that the relevant group is the one labelled #4 in the Shephard-Todd list and the rest of the assertion follows.

(c) Similarly, the point  $P_2$  corresponds for instance to  $x_0 = x_1 = x_3$  and the additional computation  $4 = (1 - (2 + 7)/12)^{-1}$  shows that the spherical triangle group associated to the projective action of the isotropy group at  $\xi \in \mathcal{T}$  is  $(2, 3, 4)$ . Indeed, we have to consider the triangle with angles  $(\pi/4, \pi/4, 2\pi/3)$  and take the symmetry into account (i.e. dividing the triangle into two parts), so that we obtain a triangle with angles  $(\pi/2, \pi/3, \pi/4)$ . The center has order given by  $12 = (1 - (2 + 2 + 7)/12)^{-1}$ . Comparing with [ST, Table 2], we see that the relevant group is the one labelled #10 in the Shephard-Todd list.

(d) Follows from the interpretation of the stabilizer of  $M \in \mathcal{M}_A$  as a central extension with center of order 3 (corresponding to the order of the reflection with mirror  $M$ ) of a Deligne-Mostow group with weights  $(2, 4, 7, 11)/12$  coming for instance from the collapsing

of  $x_1$  and  $x_2$ . The associated triangle group is  $(2, 4, 12)$  since  $2 = (1 - (2 + 4)/12)^{-1}$ ,  $4 = (1 - (2 + 7)/12)^{-1}$  and  $12 = (1 - (4 + 7)/12)^{-1}$ .

(e) Similarly, the stabilizer of  $M \in \mathcal{M}_B$  is a central extension with center of order 4 (corresponding to the order of the reflection with mirror  $M$ ) of a (Deligne-)Mostow group with weights  $(2, 2, 9, 11)/12$  coming for instance from the collapsing of  $x_2$  and  $x_3$ . We have moreover to take care of the symmetry coming from the first two weights. The associated triangle group is  $(2, 3, 12)$  since  $3/2 = (1 - (2 + 2)/12)^{-1}$  and  $12 = (1 - (2 + 9)/12)^{-1}$  so that we have to divide into two parts a triangle with angles  $(2\pi/3, \pi/12, \pi/12)$ .  $\square$

**1.3.** We come back to the description of  $B_{\mathbb{C}}^2$  and  $\bar{\Gamma}$  in the more concrete terms of §1.1. The elements  $u$  and  $v$  of  $\bar{\Gamma}$  are complex reflections of order 3 and 4, respectively. For  $\alpha \in \mathbb{C}$ , define

$$M_\alpha = \{(z, w) \in B_{\mathbb{C}}^2 : z = \alpha w\}.$$

We also let  $M_\infty = \{(z, w) \in B_{\mathbb{C}}^2 : w = 0\}$ . Setting  $c = (r-1)(\zeta^3-1)/2 = \zeta^2 - \zeta$ , one can check that  $u$  fixes each point of  $M_c$ , and  $v$  fixes each point of  $M_0$ . As a consequence of Lemma 1(a),  $\mathcal{M}_A = \{g(M_c) : g \in \bar{\Gamma}\}$  and  $\mathcal{M}_B = \{g(M_0) : g \in \bar{\Gamma}\}$ . Of course,  $g(M_c)$  and  $g(M_0)$  are the sets of points of  $B_{\mathbb{C}}^2$  fixed by the complex reflection  $gug^{-1}$ , and  $gvv^{-1}$ , respectively. For  $\xi \in B_{\mathbb{C}}^2$ , let  $\mathcal{M}_A(\xi)$ , respectively  $\mathcal{M}_B(\xi)$  denote the set of distinct mirrors  $M$ , of type A and B, respectively, containing  $\xi$ . In the following proposition, recall that  $j = (uv)^2$ .

**Proposition 1.** *The non-trivial elements of finite order in  $\bar{\Gamma}$  are all conjugate to one of the elements in the following table, or the inverse of one of these.*

| $d$ | Representatives of elements of order $d$                                                         |
|-----|--------------------------------------------------------------------------------------------------|
| 2   | $v^2, j^6, (bu^{-1})^2$                                                                          |
| 3   | $u, j^4, uj^4, buv$                                                                              |
| 4   | $v, j^3, vj^3, v^2j^3, bu^{-1}$                                                                  |
| 6   | $j^2, v^2j^2, v^2uj, v^2uj^5, bv^2u^{-1}j, bv^2$                                                 |
| 8   | $uvj, \zeta^{-1}bj, (\zeta^{-1}bj)^3$                                                            |
| 12  | $j, j^5, uv^{-1}j^2, uv^{-1}j^3, uv^{-1}j^6, uv^{-1}j^{-1}, v^2j, uv^2, uj, uj^3, bv, (bv)^{-5}$ |
| 24  | $uv, vuj^2$                                                                                      |

*Proof.* Elements of  $\bar{\Gamma}$  which fix points of  $B_{\mathbb{C}}^2$  must have finite order, because  $\bar{\Gamma}$  acts discontinuously on  $B_{\mathbb{C}}^2$ . Conversely (see [CS2, Lemma 3.3]) any element of finite order in  $\bar{\Gamma}$  fixes at least one point of  $B_{\mathbb{C}}^2$ , and is conjugate to an element of  $K \cup bK \cup bu^{-1}bK$ . One can easily list the nontrivial elements of finite order in this last set (there are 408 of them, 76 in  $bK$  and 45 in  $bu^{-1}bK$ ), all having order dividing 24. Routine calculations show that any such element (and hence each nontrivial element of finite order in  $\bar{\Gamma}$ ) has a matrix representative  $g$  conjugate to one of the elements in the above table, or its inverse.  $\square$

For  $\alpha \in \mathbb{C} \cup \{\infty\}$  and for  $\xi \in B_{\mathbb{C}}^2$ , let

$$\bar{\Gamma}_\alpha = \{g \in \bar{\Gamma} : g(M_\alpha) = M_\alpha\} \quad \text{and} \quad \bar{\Gamma}_\xi = \{g \in \bar{\Gamma} : g.\xi = \xi\}$$

denote the stabilizer of  $M_\alpha$  and  $\xi$ , respectively. In §1.2, we described the  $\xi \in B_{\mathbb{C}}^2$  for which  $\bar{\Gamma}_\xi \neq \{1\}$ . The result can be summed up in the following table, in which  $\bar{\Gamma}(\xi)$  denotes the image of  $\xi \in B_{\mathbb{C}}^2$  in  $R = \bar{\Gamma} \backslash B_{\mathbb{C}}^2$ .

| $\bar{\Gamma}(\xi)$ | $ \bar{\Gamma}_\xi $ | $ \mathcal{M}_A(\xi) $ | $ \mathcal{M}_B(\xi) $ |
|---------------------|----------------------|------------------------|------------------------|
| $P_1$               | 24                   | 4                      | 0                      |
| $P_2$               | 288                  | 8                      | 6                      |
| $P_3$               | 12                   | 1                      | 1                      |
| $P_4$               | 8                    | 0                      | 1                      |
| $P_5$               | 3                    | 0                      | 0                      |
| generic $D_A$       | 3                    | 1                      | 0                      |
| generic $D_B$       | 4                    | 0                      | 1                      |

where *generic*  $D_A$  (resp.  $D_B$ ) means that  $\bar{\Gamma}(\xi) \in D_A$  (resp.  $D_B$ ) and  $\bar{\Gamma}(\xi) \neq P_1, P_2, P_3$  (resp.  $P_2, P_3, P_4$ ). Two points of  $B_{\mathbb{C}}^2$  are particularly important: the origin  $O$ , such that  $\bar{\Gamma}(O) = P_2$  (i.e.  $O \in \mathcal{T}$ ), and

$$P = \left( \frac{c(\zeta - 1)}{\sqrt{r - 1}}, \frac{\zeta - 1}{\sqrt{r - 1}} \right), \quad (1)$$

such that  $\bar{\Gamma}(P) = P_1$  (i.e.  $P \in \mathcal{P}$ ). By [CS2, Lemma 3.1],  $\bar{\Gamma}_O = K$ . We can show that  $\bar{\Gamma}_P = \langle u, b \rangle$ , which has cardinality 24, by first noting that  $g.P = P$  implies that  $d(g.O, 0) \leq 2d(P, 0)$  (where  $d$  is the hyperbolic metric on  $B_{\mathbb{C}}^2$ ), which equals  $\cosh^{-1}(\sqrt{3} + 1)$ . This implies (see [CS2, §3]) that  $g \in K \cup K b K \cup K b u^{-1} b K$ , so we need only check which  $g$  in this finite set fix  $P$ .

Another important point will be the fixed point

$$Q = \left( \frac{c_1}{\sqrt{r - 1}}, \frac{c_2}{\sqrt{r - 1}} \right) \quad (2)$$

of  $buv$  such that  $\bar{\Gamma}(Q) = P_5$  where for  $\lambda = e^{-\pi i/18}$ ,

$$c_1 = \zeta^3 - \zeta^2 - \zeta + 1 + (\zeta^2 - \zeta + 1)\lambda + (-\zeta^3 + \zeta^2 - 1)\lambda^2, \quad \text{and} \quad c_2 = \zeta^3 - (\zeta - 1)\lambda^2.$$

We similarly check that  $\bar{\Gamma}_Q = \langle buv \rangle$ , so that  $|\bar{\Gamma}_Q| = 3$ . The points  $P_3$  and  $P_4$  in  $R$  are, respectively, the images  $\bar{\Gamma}(\xi)$  of the fixed points

$$\xi = \left( 0, \frac{\zeta - 1}{\sqrt{r - 1}} \right) \quad \text{and} \quad \xi = \left( 0, \frac{(e^{-\pi i/4}(\zeta - 1)(\zeta^2 - 1) + 1)\zeta^3}{\sqrt{r - 1}} \right),$$

of  $bv$  and  $bj$ . We similarly verify that  $\bar{\Gamma}_\xi = \langle bv \rangle$  and  $\langle bj \rangle$ , so that  $|\bar{\Gamma}_\xi| = 12$  and 8, respectively.

The following lemma adds further detail to Lemma 1(c) and is easily checked.

**Lemma 2.** *The orbit under the finite group  $K$  of  $M_c$  consists of the eight mirrors  $M_\alpha$  for  $\alpha = c_{\pm\pm\pm} = \pm(r \pm 1)(i \pm 1)/2$  (so that for example  $c = c_{+-}$ ), and  $\mathcal{M}_A(O)$  is the set of these  $M_\alpha$ 's. The 8 elements  $k_\alpha \in K$  in the table below are such that  $k_\alpha(M_c) = M_\alpha$ .*

| $\alpha$   | $c_{+++}$ | $c_{--+}$ | $c_{---}$ | $c_{+-}$ | $c_{-++}$    | $c_{-+-}$     | $c_{+++}$       | $c_{++-}$       |
|------------|-----------|-----------|-----------|----------|--------------|---------------|-----------------|-----------------|
| $k_\alpha$ | 1         | $v$       | $v^2$     | $v^3$    | $u^{-1}v^2u$ | $vu^{-1}v^2u$ | $v^2u^{-1}v^2u$ | $v^3u^{-1}v^2u$ |

The orbit under  $K$  of  $M_0$  consists of the 6 mirrors  $M_\alpha$ ,  $\alpha \in \{0, 1, -1, i, -i, \infty\}$ , and  $\mathcal{M}_B(O)$  is the set of these  $M_\alpha$ 's. The 6 elements  $k_\alpha \in K$  in the table below satisfy  $k_\alpha(M_0) = M_\alpha$ .

| $\alpha$   | 0 | $i$  | $-1$  | $-i$    | 1       | $\infty$        |
|------------|---|------|-------|---------|---------|-----------------|
| $k_\alpha$ | 1 | $uj$ | $vuj$ | $v^2uj$ | $v^3uj$ | $u^{-1}v^2uj^6$ |

**1.4.** Cartwright and Steger discovered a very interesting torsion-free subgroup  $\Pi$  of  $\bar{\Gamma}$  with finite index. The surface  $X = \Pi \backslash B_{\mathbb{C}}^2$  is called the Cartwright-Steger surface in this article.

**Theorem 2** ([CS2]). *The elements*

$$a_1 = vuv^{-1}j^4buvj^2, \quad a_2 = v^2ubuv^{-1}uv^2j \quad \text{and} \quad a_3 = u^{-1}v^2uj^9bv^{-1}uv^{-1}j^8$$

*of  $\bar{\Gamma}$  generate a torsion-free subgroup  $\Pi$  of index 864, with  $\Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$ .*

*Proof.* Using the given presentation of  $\bar{\Gamma}$ , the Magma `Index` command shows that  $\Pi$  has index 864 in  $\bar{\Gamma}$ . We see that  $\Pi$  is torsion-free as follows. The 864 elements  $b^{\mu}k$ , for  $\mu = 0, 1, -1$  and  $k \in K$ , form a set of representatives for the cosets  $\Pi g$  of  $\Pi$  in  $\bar{\Gamma}$ . One can verify this by a method we shall use repeatedly: for  $g = b^{\mu}k$  and  $g' = b^{\mu'}k'$ , we check that  $\Pi g \neq \Pi g'$  unless  $\mu' = \mu$  and  $k' = k$  by having Magma calculate the index in  $\bar{\Gamma}$  of  $\langle a_1, a_2, a_3, g'g^{-1} \rangle$ .

If  $1 \neq \pi \in \Pi$  has finite order, then  $\pi = gtg^{-1}$  for one of the elements  $t$  given in the table of Proposition 1, or the inverse of one of these. But then  $(b^{\mu}k)t(b^{\mu}k)^{-1} \in \Pi$  for some  $\mu \in \{0, 1, -1\}$  and  $k \in K$ , and Magma's `Index` command shows that this is not the case.

The Magma `AbelianQuotientInvariants` command shows that  $\Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$ . For any isomorphism  $f : \Pi/[\Pi, \Pi] \rightarrow \mathbb{Z}^2$ , the image under  $f$  of  $a_1^3a_2^{-2}a_3^7$  is trivial. We can choose  $f$  so that it maps  $a_1, a_2$  and  $a_3$  to  $(1, 3)$ ,  $(-2, 1)$  and  $(-1, -1)$ , respectively. So  $f(a_1a_2^{-1}a_3^2) = (1, 0)$  and  $f(a_1^{-1}a_2a_3^{-3}) = (0, 1)$ .  $\square$

Magma shows that the normalizer of  $\Pi$  in  $\bar{\Gamma}$  contains  $\Pi$  as a subgroup of index 3, and is generated by  $\Pi$  and  $j^4$ . One may verify that

$$\begin{aligned} j^4a_1j^{-4} &= a_3a_2^{-3}a_3^3a_1, \\ j^4a_2j^{-4} &= a_3^{-1}, \quad \text{and} \\ j^4a_3j^{-4} &= a_1^{-1}a_2^{-1}a_1a_2^2a_1^{-1}a_2^{-1}a_1a_3^{-1}a_1^{-1}a_2a_1. \end{aligned}$$

With the above isomorphism  $f : \Pi/[\Pi, \Pi] \rightarrow \mathbb{Z}^2$ ,

$$f(j^4\pi j^{-4}) = f(\pi) \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{for all } \pi \in \Pi. \quad (3)$$

**1.5.** Cartwright and Steger noticed that the group  $\Pi$  can be exhibited as a congruence subgroup of  $\bar{\Gamma}$ : we have two reductions  $r_2 : \mathbb{Z}[\zeta] \rightarrow \mathbb{F}_4 = \mathbb{F}_2[\omega]$  and  $r_3 : \mathbb{Z}[\zeta] \rightarrow \mathbb{F}_9 = \mathbb{F}_3[i]$  defined by sending  $\zeta$  to  $\omega$  (resp.  $i$ ) where  $1 + \omega + \omega^2 = 0$  (resp.  $i^2 = -1$ ). They induce (surjective) group morphisms  $\rho_2 : \bar{\Gamma} \rightarrow \text{PU}(3, \mathbb{F}_4)$  and  $\rho_3 : \bar{\Gamma} \rightarrow \text{PU}(3, \mathbb{F}_9)$  (recall that  $\text{PU}(3, \mathbb{F}_4)$  and  $\text{PU}(3, \mathbb{F}_9)$  have respective cardinality 216 and 6048).

Note that for an element of  $\text{PU}(3, \mathbb{F}_4)$ , the determinant is well defined since  $\omega^3 = 1$ . This enables us to define a (surjective) morphism  $\det_2 = \det \circ \rho_2 : \bar{\Gamma} \rightarrow \mathbb{F}_4^*$ . Let us denote the subgroup  $\det_2^{-1}(1)$  of index 3 of  $\bar{\Gamma}$  by  $\Pi_2$ .

Remark also that there exist subgroups of order 21 in  $\text{PU}(3, \mathbb{F}_9)$  (they are all conjugate) and let us denote one of them by  $G_{21}$ . Then, define  $\Pi_3 := \rho_3^{-1}(G_{21})$ : it is a subgroup of  $\bar{\Gamma}$  of index 288 = 6048/21.

So  $\Pi_2 \cap \Pi_3$  is a subgroup of  $\bar{\Gamma}$  of index 864 = 3 · 288, and one can show that it is isomorphic to  $\Pi$ . As the natural map  $X = \Pi \backslash B_{\mathbb{C}}^2 \rightarrow \bar{\Gamma} \backslash B_{\mathbb{C}}^2 = R$  obviously has degree 864, one can prove

**Lemma 3.** *The Cartwright-Steger surface  $X = \Pi \backslash B_{\mathbb{C}}^2$  has the following numerical invariants:*

$$c_1^2 = 9, \quad c_2 = 3, \quad \chi(\mathcal{O}_X) = 1, \quad q = h^{1,0} = 1, \quad p_g = h^{2,0} = 1, \quad h^{1,1} = 3.$$

*Moreover,  $H^2(X, \mathbb{Z})$  is torsion free.*

*Proof.* The orbifold  $\bar{\Gamma} \backslash B_{\mathbb{C}}^2$  has orbifold Euler characteristic  $1/288$  (see [PY] or [Sa] for instance) so that  $X$  has Euler characteristic  $c_2(X) = 3 = 864/288$ . Then, as it is a two-ball

quotient,  $c_1^2(X) = 9$  and thus its arithmetic genus is  $\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2) = 1$ . Since  $\Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$ , we have  $b_1 = 2q = 2$ . So, from

$$\begin{aligned} 1 &= \chi(\mathcal{O}_X) = 1 - q + p_g, \\ 3 &= c_2(X) = 2b_0 - 2b_1 + b_2, \end{aligned}$$

we deduce that  $p_g = 1$ ,  $b_2 = 5$ , and finally,  $h^{1,1} = 3$ .

That  $H^2(X, \mathbb{Z})$  is torsion free follows immediately from the universal coefficient theorem and the fact that  $H_1(X, \mathbb{Z}) \cong \Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$  is also torsion free.  $\square$

We will see later (Corollary 1) that the Picard number of  $X$  is actually 3. It is our purpose to understand the geometric properties of the surface  $X$ , especially using its Albanese map.

## 2. CONFIGURATIONS OF SOME TOTALLY GEODESIC DIVISORS

Here we describe results about configuration of totally geodesic divisors on the Cartwright-Steger surface  $X = \Pi \backslash B_{\mathbb{C}}^2$ .

Let  $\varpi : X \rightarrow R = \bar{\Gamma} \backslash B_{\mathbb{C}}^2$  be the projection. We use the notation of §1.2. From the description of the local groups at  $P_1$ ,  $P_2$  and  $P_3$ , we know that  $\varpi^{-1}(P_2)$  consists of  $3 = 864/288$  points  $O_1 = \Pi(O)$ ,  $O_2 = \Pi(b \cdot O)$ ,  $O_3 = \Pi(b^{-1} \cdot O)$  on  $X$ ,  $\varpi^{-1}(P_1)$  consists of  $36 = 864/24$  points, and  $\varpi^{-1}(P_3)$  consists of  $72 = 864/12$  points. It is easy using Magma to find  $k_i \in K$  such that  $\varpi^{-1}(P_1) = \{\Pi(k_i \cdot P) : 1 \leq i \leq 36\}$ .

For the curves  $D_A$  and  $D_B$ , their preimages  $\varpi^{-1}(D_A)$  and  $\varpi^{-1}(D_B)$  consist of singular totally geodesic curves on  $X$ , denoted to be of types  $A$  and  $B$  respectively. By the description of  $R$  in Figure 1, the curves can only have crossings at  $\varpi^{-1}(P_i)$  for  $i = 1, 2, 3$ , and since these curves are totally geodesic, these crossings are simple. It will be crucial for us to know the genus of the irreducible components of these totally geodesic curves, as well as their singularities and the way they meet each other. In this section, we explain how we can achieve this, using computer calculations.

**2.1.** Our first step is to describe the groups  $\bar{\Gamma}_0$  and  $\bar{\Gamma}_c$  of elements stabilizing  $M_0$  and  $M_c$ , respectively.

As we saw in Lemma 1(d),  $\bar{\Gamma}_0$  is a central extension of a  $(2, 3, 12)$ -triangle group, with center of order 4. One may show that a presentation of  $\bar{\Gamma}_0$  is given by the generators  $s_2 = (jb)^{-1}$ ,  $s_3 = b$ ,  $s_{12} = j$  and  $z_0 = v$  and the relations

$$s_{12}^{12} = s_3^3 = 1, s_2^2 = z_0^3, z_0^4 = [s_{12}, z_0] = [s_3, z_0] = [s_2, z_0] = s_{12}s_3s_2 = 1.$$

We saw in Lemma 1(e) that  $\bar{\Gamma}_c$  is a central extension of a  $(2, 4, 12)$ -triangle group, with center of order 3. One may similarly show that a presentation of  $\bar{\Gamma}_c$  is given by the generators  $t_2 = (bu^{-1})^2$ ,  $t_4 = j^{-1}(bu^{-1})^2$ ,  $t_{12} = j$  and  $z_c = u$  and the relations

$$t_{12}^{12} = 1, t_4^4 = z_c, t_2^2 = z_c^3 = [t_{12}, z_c] = [t_4, z_c] = [t_2, z_c] = t_{12}t_4t_2 = 1.$$

The proof of both these facts involves somewhat lengthy calculations with explicit matrices. We give these calculations in the Magma file we shall be posting.

**2.2.** Let  $\varphi : B_{\mathbb{C}}^2 \rightarrow X$  be the natural map. If  $M$  is a mirror of type  $A$  or  $B$ , let  $\bar{\Gamma}_M$  denote the stabilizer of  $M$  (so  $\bar{\Gamma}_\alpha = \bar{\Gamma}_{M_\alpha}$ ). The group  $\Pi_M = \{\pi \in \Pi : \pi(M) = M\} = \Pi \cap \bar{\Gamma}_M$  acts on  $M$ , and is the fundamental group of the smooth curve  $\Pi_M \backslash M$ . The embedding  $M \hookrightarrow B_{\mathbb{C}}^2$  induces an immersion  $\varphi_M : \Pi_M \backslash M \rightarrow X$ . We write  $\Pi_\alpha$  instead of  $\Pi_{M_\alpha}$ . We need now to describe  $\Pi_M$ , and we start by the simpler case of mirrors of type  $B$ .

**2.3. The groups  $\Pi_M$  when  $M$  is a mirror of type  $B$ .** First, we consider  $\Pi_0 = \Pi_{M_0} = \{\pi \in \Pi : \pi(M_0) = M_0\} = \Pi \cap \bar{\Gamma}_0 = \Pi_2 \cap \Pi_3 \cap \bar{\Gamma}_0 = \det_2^{-1}(1) \cap \rho_3^{-1}(G_{21}) \cap \bar{\Gamma}_0$  by §1.5. Restricting  $\det_2$  and  $\rho_3$  to  $\bar{\Gamma}_0$ , Magma finds that  $\Pi_0$  has index 288 in  $\bar{\Gamma}_0$ .

**Proposition 2.** *The group  $\Pi_0$  has a presentation*

$$\langle u_1, \dots, u_4, v_1, \dots, v_4 : [u_1, v_1][u_2, v_2][u_3, v_3][u_4, v_4] = 1 \rangle, \quad (4)$$

with generators  $u_i, v_i$ , given below, and so  $\Pi_0 \backslash M_0$  is a smooth curve of genus 4.

*Proof.* As  $j^4$  normalizes  $\Pi$ , we can define  $g_1, \dots, g_8 \in \Pi$  by setting  $g_1 = a_3^{-3}a_1^{-1}a_2a_1$ ,  $g_3 = a_2a_1^{-2}a_3^{-3}a_1^{-1}$ ,  $g_5 = j^4a_2a_1j^8a_2^{-1}a_3^3a_1^2$ , and  $g_7 = j^4a_1^{-1}a_2^{-1}j^4a_2a_1j^4$ , and then  $g_{2\nu} = j^4g_{2\nu-1}j^{-4}$  for  $\nu = 1, 2, 3, 4$ . These are in  $\bar{\Gamma}_0$ . Magma verifies that  $G = \langle g_1, \dots, g_8 \rangle$  has index 288 in  $\bar{\Gamma}_0$ , and so  $G = \Pi_0$ , and gives a presentation of  $\Pi_0$  which has just one relation:

$$g_1g_2g_3g_4g_5g_6g_7g_8g_1^{-1}g_3^{-1}g_5^{-1}g_7^{-1}g_2^{-1}g_4^{-1}g_6^{-1}g_8^{-1} = 1. \quad (5)$$

By a method shown to us by Jonathan Hillman, we replace the generators  $g_i$  by generators  $u_i$  and  $v_i$ , where  $u_i = \epsilon_1 \cdots \epsilon_{i-1} \delta_i \epsilon_{i-1}^{-1} \cdots \epsilon_1^{-1}$  and  $v_i = \epsilon_1 \cdots \epsilon_{i-1} \epsilon_i \epsilon_{i-1}^{-1} \cdots \epsilon_1^{-1}$ , where

$$\begin{aligned} \delta_1 &= g_1g_2g_3g_4g_5g_6g_7, & \delta_3 &= g_1, & \epsilon_1 &= g_8g_1^{-1}g_3^{-1}g_5^{-1}, & \epsilon_3 &= g_2g_3g_6^{-1}, \\ \delta_2 &= g_1g_2g_3g_4, & \delta_4 &= g_3^{-1}, & \text{and} & & \epsilon_2 &= g_5g_6g_2^{-1}, & \epsilon_4 &= g_6 \end{aligned}$$

and these generators  $u_i$  and  $v_i$  satisfy the stated relation.  $\square$

We now consider  $\Pi_M$  for the other mirrors  $M$  of type  $B$ .

**Proposition 3.** *If  $g \in \bar{\Gamma}$  and  $M = g(M_0)$  is a mirror of type  $B$ , then*

- (a) *There is a  $\pi \in \Pi$  such that  $\pi(M) = M_0, M_1$  or  $M_\infty$ .*
- (b) *Correspondingly,  $\Pi_M$  is conjugate in  $\Pi$  to either  $\Pi_0, \Pi_1$  or  $\Pi_\infty$ .*
- (c)  *$\Pi_M = g\Pi_0g^{-1}$ .*
- (d)  *$h(\Pi_M)h^{-1} = \Pi_{h(M)}$  for any  $h \in \bar{\Gamma}$ .*
- (e) *The three possibilities in (a) are mutually exclusive.*

In particular, it follows from (c) that for any mirror  $M$  of type  $B$ ,  $\Pi_M \backslash M \cong \Pi_0 \backslash M_0$ .

*Proof.* (a) The elements  $b^\mu k$ ,  $\mu = 0, 1, -1$  and  $k \in K$ , form a set of coset representatives of  $\Pi$  in  $\bar{\Gamma}$ . So using Lemma 2, we may assume that  $M = b^\mu(M_\alpha)$  for some  $\mu \in \{0, 1, -1\}$  and  $\alpha \in \{0, \pm 1, \pm i, \infty\}$ . Then, searching amongst short words in the generators  $a_i$  of  $\Pi$ , we quickly find  $\pi \in \Pi$  such that  $\pi(M) = M_\beta$  for  $\beta \in \{0, 1, \infty\}$ . For example, taking  $\pi = a_3^3a_1^2a_2^{-1}$ , we have  $\pi(bM_{-1}) = M_1$ . This proves (a), and (b) follows immediately.

(c) We first show that  $h\Pi_0h^{-1} \subset \Pi$  for each  $h \in \bar{\Gamma}$ . We may assume that  $h = b^\mu k$  as in (a). For each of the 8 generators  $g_j$  of  $\Pi_0$  given in the proof of Proposition 2 we have Magma check that  $\langle a_1, a_2, a_3, hg_jh^{-1} \rangle$  has index 864 in  $\bar{\Gamma}$ , so that  $hg_jh^{-1} \in \Pi$ . It follows, in particular, that  $h\Pi_0h^{-1} = \Pi_0$  for each  $h \in \bar{\Gamma}_0$ . We next prove (c) in the cases  $g = k_\beta$ ,  $\beta = 1, \infty$ . Now  $k_\beta\Pi_0k_\beta^{-1} \subset \Pi$  and so  $\Pi_0 \subset k_\beta^{-1}\Pi_\beta k_\beta \subset \bar{\Gamma}_0$ . Choose a transversal  $t_1, \dots, t_{288}$  of  $\Pi_0$  in  $\bar{\Gamma}_0$ , i.e., a set of elements  $t_i$  such that  $\bar{\Gamma}_0$  is the union of the distinct cosets  $\Pi_0 t_i$ . Then Magma verifies that the index in  $\bar{\Gamma}$  of  $\langle a_1, a_2, a_3, k_\beta t_i k_\beta^{-1} \rangle$  is less than 864 if  $i \neq 1$ . Thus  $\Pi_0 = k_\beta^{-1}\Pi_\beta k_\beta$ , and (c) holds for  $g = k_\beta$ ,  $\beta = 0, 1, \infty$ . By (a), for our given  $g$ , there is a  $\pi \in \Pi$  so that  $g(M_0) = \pi(M_\beta)$  for one of these  $\beta$ 's. Then  $h = k_\beta^{-1}\pi^{-1}g$  is in  $\bar{\Gamma}_0$ , so that  $h\Pi_0h^{-1} = \Pi_0$ . Then  $(\pi^{-1}g)\Pi_0(\pi^{-1}g)^{-1} = \Pi_\beta$  by (c) for  $g = k_\beta$ . Thus  $g\Pi_0g^{-1} = \pi(\Pi_\beta)\pi^{-1} = \Pi_{\pi(M_\beta)} = \Pi_M$ . Part (d) is immediate from (c).

(e) is a consequence of the discussion in §2.5 as well as Proposition 6 below.  $\square$

If  $M$  is a mirror of type  $B$ , then by Proposition 3(a), the image of the immersion  $\varphi_M : \Pi_M \backslash M \rightarrow X$  is equal to the image of  $\varphi_{M'}$  for  $M' = M_0, M_\infty$ , or  $M_1$ . We will denote by  $E_1, E_2$  and  $E_3$  respectively these images (which are distinct by Proposition 3(e)). Notice that the curves  $E_i$  are singular since they self-intersect (see Proposition 6) and that the maps  $\varphi_M$  are therefore the normalization maps.

To calculate entries in the table in §2.8, we need explicit generators for  $\Pi_\infty$ . We start with the generators  $g_i'' = k_\infty g_i k_\infty^{-1}$ , where  $g_1, \dots, g_8$  are as in proof of Proposition 2. The  $g_i''$  satisfy exactly the same relation as do the  $g_i$ 's, and so standard generators  $u_i$  and  $v_i$  can be found for  $\Pi_\infty$  in exactly the same way as was done for  $\Pi_0$ . To calculate the  $f(u_i)$  and  $f(v_i)$ 's (with  $f$  as in (3)), we need to express the  $g_i''$ 's in terms of the generators of  $\Pi$ . One may verify that:

$$\begin{aligned} g_1'' &= j^4(a_1^{-1}a_3^{-2}a_1^{-1})j^8a_1^{-1}a_2^{-1}, & g_5'' &= j^8(a_2^{-1}a_3^{-1})j^4, \\ g_3'' &= j^8(a_3a_1a_2a_1^{-1}a_2^{-1})j^4, & g_7'' &= j^4(a_1a_3a_1^{-1}a_3^{-2})j^8, \end{aligned}$$

and  $g_{2\nu}'' = j^4g_{2\nu-1}''j^8$  for  $\nu = 1, 2, 3, 4$ .

**2.4. The groups  $\Pi_M$  when  $M$  is a mirror of type  $A$ .** Magma finds that  $\Pi_c$  has index 324 in  $\bar{\Gamma}_c$ .

**Proposition 4.** *The group  $\Pi_c$  has a presentation*

$$\langle u_1, \dots, u_{10}, v_1, \dots, v_{10} : [u_1, v_1][u_2, v_2] \cdots [u_9, v_9][u_{10}, v_{10}] = 1 \rangle, \quad (6)$$

and so  $\Pi_c \backslash M_c$  is a smooth curve of genus 10.

*Proof.* The proof is very similar to that of Proposition 2. We define 20 elements  $g_1, \dots, g_{20}$  of  $\Pi$  by setting

$$\begin{aligned} g_1 &= j^8a_1^{-1}a_2a_1a_3a_1^{-1}j^4a_2a_1, & g_{12} &= a_2^{-1}a_1a_3a_1^{-1}a_3^{-1}j^4a_3a_1a_2^2a_1^{-1}a_2^{-1}j^8, \\ g_3 &= j^4a_2a_1a_2^{-2}a_1^{-1}a_3j^4a_3^3j^4, & g_{15} &= j^4a_1j^4a_2a_3a_1^{-1}j^4, \\ g_5 &= j^8a_1^{-1}j^4a_2a_1j^4a_3a_2^{-1}a_1a_3a_1^{-1}j^8, & g_{17} &= j^8a_1^{-2}a_2^{-1}j^4a_3a_1a_2a_1, \\ g_7 &= j^8a_2a_1j^4a_3^{-1}j^4a_2a_1^{-1}a_2^{-1}a_3^{-3}j^8, & g_{19} &= a_2^{-1}a_1a_3a_1^{-1}a_3^{-2}j^4a_1a_2j^4a_1^{-1}a_2^{-1}j^4, \\ g_9 &= j^8a_1^{-1}a_2^{-2}a_1^{-1}a_3^{-1}j^8a_1^{-1}a_2^{-1}j^8, \end{aligned}$$

and also  $g_{\nu+1} = j^4g_\nu j^{-4}$  for  $\nu \in \{1, 3, 5, 7, 9, 10, 12, 13, 15, 17, 19\}$ . These are in  $\bar{\Gamma}_c$ . Magma verifies that  $G = \langle g_1, \dots, g_{20} \rangle$  has index 324 in  $\bar{\Gamma}_c$ , and so  $G = \Pi_c$ , and gives a presentation of  $\Pi_c$  which has just one relation:

$$\begin{aligned} &g_4g_{14}^{-1}g_2^{-1}g_{17}^{-1}g_9g_{19}g_{20}g_{14}g_7^{-1}g_{10}^{-1}g_5^{-1}g_{16}^{-1}g_3^{-1}g_{12}^{-1}g_1g_2g_{18}^{-1}g_{10}g_{19}^{-1}g_{12} \\ &\times g_8^{-1}g_{11}^{-1}g_6^{-1}g_{15}g_{16}g_4^{-1}g_{13}^{-1}g_1^{-1}g_{17}g_{18}g_{11}g_{20}^{-1}g_{13}g_7g_8g_9^{-1}g_5g_6g_{15}^{-1}g_3 = 1. \end{aligned}$$

Using the same method as in the proof of Proposition 2, we can replace the generators  $g_i$  by generators  $u_i$  and  $v_i$  satisfying the given relation. We omit the details.  $\square$

We now consider  $\Pi_M$  for the other mirrors  $M$  of type  $A$ . As well as  $c = c_{+-}$ , the parameter  $-c = c_{--}$  is important in the next result.

**Proposition 5.** *If  $g \in \bar{\Gamma}$  and  $M = g(M_c)$  is a mirror of type  $A$ , then*

- (a) *There is a  $\pi \in \Pi$  such that  $\pi(M) = M'$ , where  $M' \in \{M_c, M_{-c}, b(M_c), b^{-1}(M_c)\}$ .*
- (b) *If  $M'$  is as in (a), then  $\Pi_M$  is conjugate in  $\Pi$  to  $\Pi_{M'}$ .*
- (c)  *$\Pi_M = g\Pi_c g^{-1}$  in the first two cases of (a), and in particular if  $g = k_\alpha$  for any  $\alpha \in \{c_{+++}, \dots, c_{---}\}$ , so that  $\Pi_\alpha = k_\alpha \Pi_c k_\alpha^{-1}$  for all these  $\alpha$ 's.*
- (d) *In the other two cases of (a),  $g\Pi_c g^{-1}$  has index 3 in  $\Pi_M$ .*
- (e) *The four possibilities in (a) are mutually exclusive.*

*Proof.* The proof is similar to that of Proposition 3. For (a), we may assume that  $M = b^\mu(M_\alpha)$  for some  $\mu \in \{0, 1, -1\}$  and  $\alpha \in \{c_{+++}, \dots, c_{---}\}$ . For each of these  $M$ 's, we find explicit  $\pi \in \Pi$  such that  $\pi(M) = M'$  for an  $M'$  in the given list. The most complicated  $\pi$  needed is  $\pi = a_2a_1^{-2}a_3^{-1}a_1a_3^{-1}a_1^{-1}a_2^{-2}$ , satisfying  $\pi(b^{-1}(M_{c_{+-}})) = b(M_c)$ .

In proving (c) and (d), we first show that  $h\Pi_c h^{-1} \subset \Pi$  for all  $h \in \bar{\Gamma}$  as in Proposition 3, and therefore that  $h\Pi_c h^{-1} = \Pi_c$  for  $h \in \bar{\Gamma}_c$ . We are reduced to proving (c) and (d) for  $g = k_c, k_{-c}, b$  and  $b^{-1}$ . We have  $\Pi_c \subset g^{-1}\Pi_M g \subset \bar{\Gamma}_c$ , and choose a transversal  $t_1 = 1, \dots, t_{324}$  of  $\Pi_c$

in  $\bar{\Gamma}_c$ . Magma verifies that  $gt_i g^{-1} \in \Pi$  only for  $i = 1$  in the first two cases as in Proposition 3, but that  $gt_i g^{-1} \in \Pi$  for three  $i$ 's in the last two cases.

Finally, (e) is a consequence of the discussion in §2.5 as well as Proposition 7 below.  $\square$

In Proposition 5(d),  $\Pi_M \backslash M$  is a smooth curve of genus 4 by the Riemann-Hurwitz formula, and we can find explicit generators  $u_i, v_i$  of  $\Pi_M$  such that  $[u_1, v_1][u_2, v_2][u_3, v_3][u_4, v_4] = 1$ . When  $M = b(M_c)$ , the following eight elements generate  $\Pi_M$ :

$$\begin{aligned} p_1 &= a_2^3 a_1^{-1} a_3^{-1} j^8 a_2^{-2} a_1^{-1} j^4, & p_5 &= a_3^3 a_1 a_2^2 j^4 a_1^{-1} j^8 a_3^2 a_1 a_2^{-3}, \\ p_2 &= a_3^3 a_1 a_2^2 a_1 j^4 a_3^{-1} j^8 a_3^{-2} a_1^{-1} a_3^{-3}, & p_6 &= a_3^3 a_1 a_2 a_1 a_3 a_2^{-3}, \\ p_3 &= j^8 a_1^{-1} a_3^{-3} a_2^2 j^4 a_3^{-2} a_1^{-1} a_3^{-3}, & p_7 &= a_3^3 a_1 j^8 a_1 a_2^{-2} a_1^{-1} a_3^2 j^4, \\ p_4 &= j^8 a_2 a_1 a_2^{-2} a_1^{-1} j^4 a_3^3 a_1^2 a_2^{-1}, & p_8 &= j^4 a_3^{-2} j^8 a_2 a_1 a_2 a_1 a_2^{-2}, \end{aligned}$$

and satisfy the single relation

$$p_5^{-1} p_2^{-1} p_5 p_1 p_3 p_8^{-1} p_4 p_1^{-1} p_7^{-1} p_6^{-1} p_7 p_2 p_3^{-1} p_8 p_4^{-1} p_6 = 1.$$

Following the same procedure as in the proof of Proposition 2, we obtain a presentation (4) for  $\Pi_M$ , with  $u_i = \epsilon_1 \cdots \epsilon_{i-1} \delta_i \epsilon_{i-1}^{-1} \cdots \epsilon_1^{-1}$  and  $v_i = \epsilon_1 \cdots \epsilon_{i-1} \epsilon_i \epsilon_{i-1}^{-1} \cdots \epsilon_1^{-1}$  for

$$\begin{aligned} \delta_1 &= p_5^{-1} p_2^{-1} p_5 p_1 p_3 p_8^{-1} p_4 p_1^{-1} p_7^{-1}, & \delta_3 &= p_5^{-1} p_2^{-1} p_5 p_1, & \text{and} & & \epsilon_1 &= p_6^{-1}, & \epsilon_3 &= p_3, \\ \delta_2 &= p_5^{-1} p_2^{-1} p_5 p_1 p_3 p_8^{-1}, & \delta_4 &= p_5^{-1}, & & & \epsilon_2 &= p_4 p_1^{-1} p_2 p_3^{-1}, & \epsilon_4 &= p_2^{-1}. \end{aligned}$$

If  $M$  is a mirror of type  $A$ , then by Proposition 5(a), the image of the immersion  $\varphi_M : \Pi_M \backslash M \rightarrow X$  is equal to the image of  $\varphi_{M'}$  for  $M' = b(M_c), b^{-1}(M_c), M_c$ , or  $M_{-c}$ . We will denote by  $C_1, C_2, C_3$  and  $C_4$  respectively these images. They are distinct by Proposition 5(e), singular, and the maps  $\varphi_M$  are the normalization maps.

**2.5.** In §2.3 and §2.4, we have identified 7 distinct irreducible totally geodesic curves in  $X$ , 4 of type  $A$ , and 3 of type  $B$ . Just the knowledge of the indices of the groups  $\Pi_M$  in  $\bar{\Gamma}_M$  together with Lemma 1(d) and (e) enables us to determine the genus of the curves  $\Pi_M \backslash M$ .

For instance, since  $\Pi_M$  has index 288 in  $\bar{\Gamma}_M$  when  $M$  is of type  $B$ , and since the center of  $\bar{\Gamma}_M$  (which acts trivially on  $M$ ) has order 4, the normalization  $\hat{E}_i$  of the curve  $E_i$  is an orbifold covering of degree  $72 = 288/4$  of the orbifold  $D_B \cong \mathbb{P}_{\mathbb{C}}^1$  endowed with three orbifold points  $(P_4, P_3, P_2)$  of respective multiplicities  $(2, 3, 12)$  hence by the Riemann-Hurwitz formula, its genus is indeed

$$g(\hat{E}_i) = \frac{72}{2} \left( -2 + \frac{2-1}{2} + \frac{3-1}{3} + \frac{12-1}{12} \right) + 1 = 4.$$

Note that  $864 = 4 \cdot 3 \cdot 72$ , where 4 is the order of the reflections of type  $B$  and 3 the number of curves of type  $B$ , which proves Proposition 3(e).

In the same way, the normalizations of  $C_1$  and  $C_2$  (resp.  $C_3$  and  $C_4$ ) are orbifold coverings of degree 36 (resp. 108) of the orbifold  $D_A$  whose normalization is  $\mathbb{P}_{\mathbb{C}}^1$ , endowed with three orbifold points  $(P_1, P_3, P_2)$  of respective multiplicities  $(2, 4, 12)$  so that  $g(\hat{C}_1) = g(\hat{C}_2) = 4$  and  $g(\hat{C}_3) = g(\hat{C}_4) = 10$ . Here again,  $864 = 3(2 \cdot 36 + 2 \cdot 108)$  where 3 is the order of the reflections of type  $A$ , which proves Proposition 5(e).

However, we will need to know explicit generators of the various groups  $\Pi_M$  (see below).

**2.6.** Now, we want to find out the singularities of the curves  $C_i$  and  $E_i$ . The next result is a straightforward consequence of the discussion at the beginning of §2.

**Lemma 4.** *Suppose that  $x \in X$  is the image under  $\varphi_M$  of two or more distinct elements of  $\Pi_M \backslash M$ . If  $M$  is of type  $B$ , then  $x$  must be one of the three points  $\Pi(O)$ ,  $\Pi(b.O)$  and  $\Pi(b^{-1}.O)$ . If  $M$  is of type  $A$ , then  $x$  is either one of these three points or one of the 36 points  $\Pi(k_i.P)$ , where the  $k_i$  are as above. If  $\xi \in M$ , then  $\varphi_M(\Pi_M \xi)$  is one of the three points  $\Pi(b^\mu.O)$ ,  $\mu = 0, 1, -1$ , if and only if  $\xi$  is in the  $\bar{\Gamma}$ -orbit of  $O$ , and it is one of the 36 points  $\Pi(k_i.P)$  if and only if  $\xi$  is in the  $\bar{\Gamma}$ -orbit of  $P$ .*

- Lemma 5.** (i) For each mirror  $M$  of type  $B$ , there are exactly six distinct  $\Pi_M \xi \in \Pi_M \backslash M$  such that  $\xi \in M$  is in the  $\bar{\Gamma}$ -orbit of  $O$ .
- (ii) Suppose that  $M$  is a mirror of type  $A$ , and that there is a  $\pi \in \Pi$  such that  $\pi(M) = M_c$  or  $M_{-c}$ , respectively such that  $\pi(M) = b(M_c)$  or  $b^{-1}(M_c)$ . There are exactly 9 (respectively 3) distinct  $\Pi_M \xi \in \Pi_M \backslash M$  such that  $\xi \in M$  is in the  $\bar{\Gamma}$ -orbit of  $O$ . There are exactly 54 (respectively 18) distinct  $\Pi_M \xi \in \Pi_M \backslash M$  such that  $\xi \in M$  is in the  $\bar{\Gamma}$ -orbit of  $P$ .

*Proof.* (i) This follows from the description given in §2.5. Indeed, the orbifold point  $P_2$  on  $D_B$  has weight 12 so that it has  $6 = 72/12$  preimages in  $\hat{E}_i$ .

(ii) In the same way, the orbifold point  $P_2$  on  $D_A$  has weight 12 so that it has  $9 = 108/12$  preimages in  $\hat{C}_3$  and  $\hat{C}_4$  (resp.  $3 = 36/12$  preimages in  $\hat{C}_1$  and  $\hat{C}_2$ ). Also, the orbifold point  $P_1$  has weight 2 so that it has  $54 = 108/2$  preimages in  $\hat{C}_3$  and  $\hat{C}_4$  (resp.  $18 = 36/2$  preimages in  $\hat{C}_1$  and  $\hat{C}_2$ ).  $\square$

For any mirror  $M$ , and any  $\mu \in \{0, 1, -1\}$ , let

$$n_\mu(M) = \#\{\Pi_M \xi \in \Pi_M \backslash M : \varphi_M(\Pi_M \xi) = \Pi(b^\mu \cdot O)\}.$$

By Lemma 5(i),  $n_0(M) + n_1(M) + n_{-1}(M) = 6$  if  $M$  is of type  $B$ .

**Proposition 6.** If  $M$  is a mirror of type  $B$ , then according to the three possibilities in Proposition 3(a),  $(n_0(M), n_1(M), n_{-1}(M))$  is  $(3, 1, 2)$ ,  $(1, 4, 1)$  and  $(2, 1, 3)$ , respectively.

*Proof.* Now  $f : \Pi_M g(K \cap \bar{\Gamma}_M) \mapsto \Pi_M(g(0))$  maps  $\Pi_M \backslash \bar{\Gamma}_M / (K \cap \bar{\Gamma}_M)$  injectively into the subset of six elements of  $\Pi_M \backslash M$  of Lemma 5(i). By Proposition 3, we may suppose that  $M = M_0, M_1$  or  $M_\infty$ . When  $M = M_0$ , Magma verifies that the following six  $g$ 's,

$$1, j^9 b j^7, j^{10} b j^7, j^{11} b j^7, b j^7 \text{ and } j^9 b j^2 b^{-1} j,$$

are in  $\bar{\Gamma}_M$  and belong to different double cosets. So  $f$  is a bijection, and  $n_\mu(M_0)$  is the number of these  $g$ 's such that  $g \in \Pi b^\mu K$ . We use Magma's `Index` routine to check when  $g(b^\mu k)^{-1} \in \Pi$  for some  $k \in K$ . For  $M = M_\alpha$ ,  $\alpha = 1$  and  $\infty$ , we replace these  $g$ 's by  $k_\alpha g k_\alpha^{-1}$ , for  $k_\alpha$  as in Lemma 2.  $\square$

That  $n_0(M_0), n_0(M_1), n_0(M_\infty)$  are distinct gives another proof that the images of  $\varphi_{M_0}, \varphi_{M_1}$  and  $\varphi_{M_\infty}$  are distinct, which is equivalent to Proposition 3(e).

We now calculate  $n_\nu(M)$ ,  $\nu = 0, 1, -1$ , for mirrors  $M$  of type  $A$ , as well as the numbers

$$m_i(M) = \#\{\Pi_M \xi \in \Pi_M \backslash M : \varphi_M(\Pi_M \xi) = \Pi(k_i \cdot P)\}$$

for  $i = 1, \dots, 36$  (recall that the  $k_i$ 's were defined at the beginning of §2). If  $\pi \in \Pi$  and  $M' = \pi(M)$ , then  $n_\nu(M') = n_\nu(M)$  and  $m_i(M') = m_i(M)$  for each  $\nu$  and  $i$ , and so by Proposition 5(a), we need only do the calculation for  $M_c, M_{-c}, b(M_c)$  and  $b^{-1}(M_c)$ .

**Proposition 7.** For mirrors  $M$  of type  $A$ ,  $(n_0(M), n_1(M), n_{-1}(M))$  is  $(4, 3, 2)$  for the first two cases in Proposition 5(a), and  $(0, 1, 2)$  for the other two.

For a suitable ordering of the  $k_i$ , the numbers  $m_i = m_i(M)$  are as follows:

| $M$           | $m_1, \dots, m_{12}$ | $m_{13}, \dots, m_{18}$ | $m_{19}, \dots, m_{24}$ | $m_{25}, \dots, m_{36}$ |
|---------------|----------------------|-------------------------|-------------------------|-------------------------|
| $M_c$         | 2                    | 0                       | 3                       | 1                       |
| $M_{-c}$      | 2                    | 3                       | 0                       | 1                       |
| $b(M_c)$      | 0                    | 0                       | 1                       | 1                       |
| $b^{-1}(M_c)$ | 0                    | 1                       | 0                       | 1                       |

*Proof.* As in Proposition 6, to get the numbers  $n_\nu(M)$ , we choose representatives  $\gamma \in \bar{\Gamma}_M$  of the 9 (resp. 3) distinct double cosets  $\Pi_M \gamma(\bar{\Gamma}_M \cap K)$  (respectively,  $\Pi_M \gamma(\bar{\Gamma}_M \cap b^j K b^{-j})$ ) for  $M = M_c$  and  $M_{-c}$  (respectively,  $M = b^j(M_c)$ ,  $j = 1, -1$ ) and then compute their images  $\Pi \gamma K$  (respectively,  $\Pi \gamma b^j K$ ) in  $\Pi \backslash \bar{\Gamma} / K$ .

To compute the numbers  $m_i(M)$  for  $M = M_\alpha$ ,  $\alpha = c, -c$ , (respectively,  $M = b^j(M_c)$ ,  $j = 1, -1$ ), we choose representatives  $\gamma \in \bar{\Gamma}_M$  of the 54 (respectively, 18) distinct double cosets  $\Pi_M \gamma (\bar{\Gamma}_M \cap k_\alpha \bar{\Gamma}_P k_\alpha^{-1})$ , (respectively,  $\Pi_M \gamma (\bar{\Gamma}_M \cap b^j \bar{\Gamma}_P b^{-j})$ ) and compute their images  $\Pi \gamma k_\alpha \bar{\Gamma}_P$  (respectively,  $\Pi \gamma b^j \bar{\Gamma}_P$ ) in  $\Pi \backslash \bar{\Gamma} / \bar{\Gamma}_P$ .  $\square$

That  $m_i(M_c)$ ,  $m_i(M_{-c})$ ,  $m_i(b(M_c))$ ,  $m_i(b^{-1}(M_c))$  are distinct gives another proof that the images of  $\varphi_{M_c}$ ,  $\varphi_{M_{-c}}$ ,  $\varphi_{b(M_c)}$  and  $\varphi_{b^{-1}(M_c)}$  are distinct, which is equivalent to Proposition 5(e).

**2.7.** The knowledge of the numbers  $n_\nu(M)$  and  $m_i(M)$  determines the singularities of our seven totally geodesic curves. In order to determine how two distinct such curves intersect, we also need to know which of the 72 points of  $\varpi^{-1}(P_3)$  each of them contains. Using exactly the same method as in Propositions 6 and 7, we obtain

**Proposition 8.** *There are exactly 72 distinct points in  $\varpi^{-1}(P_3)$ . The set of these points may be partitioned into three subsets of size 24, consisting of the points in the images of  $M_0$ ,  $M_1$  and  $M_\infty$ , respectively. For  $\alpha = 0, 1, \infty$ , the set of 24 points belonging to the image of  $M_\alpha$  is partitioned into sets of  $n_1, n_2, n_3$  and  $n_4$  points in the images of  $M_c, M_{-c}, b(M_c)$  and  $b^{-1}(M_c)$ , respectively, where  $(n_1, n_2, n_3, n_4) = (6, 6, 6, 6)$  for  $\alpha = 0$ ,  $(n_1, n_2, n_3, n_4) = (9, 9, 3, 3)$  for  $\alpha = 1$ , and  $(n_1, n_2, n_3, n_4) = (12, 12, 0, 0)$  for  $\alpha = \infty$ .*

**2.8.** We have seen (cf. §1.4) that  $H_1(X, \mathbb{Z}) = \mathbb{Z}e_1 + \mathbb{Z}e_2 \cong \mathbb{Z}^2$  in terms of a basis  $e_1$  and  $e_2$ . For each of the genus 4 curves  $D = E_i$ ,  $i = 1, 2, 3$ , and  $D = C_j$ ,  $j = 1, 2$ , a presentation (4) can be given for  $\pi_1(\hat{D})$ . Abusing notation, we denote by  $f : H_1(\hat{D}, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z}) \cong \mathbb{Z}^2$  the homomorphism induced by the normalization of the immersed image of  $D$  in  $X$ . For  $E_1, E_2$  and  $C_1$  (which is all we need for later computations, especially in the proof of Lemma 10) we have given generators  $u_i, v_i$ ,  $i = 1, \dots, 4$ , of  $\pi_1(\hat{D})$  explicitly as words in the generators  $a_1, a_2$  and  $a_3$  of  $\Pi$ . So it is routine to compute their images  $f(u_i), f(v_j)$  in  $H_1(X, \mathbb{Z})$  in terms of  $e_1, e_2$ . We obtain:

| $D$   | $f(u_1)$   | $f(v_1)$  | $f(u_2)$  | $f(v_2)$ | $f(u_3)$  | $f(v_3)$  | $f(u_4)$  | $f(v_4)$   |
|-------|------------|-----------|-----------|----------|-----------|-----------|-----------|------------|
| $E_1$ | $(-5, -2)$ | $(-2, 7)$ | $(-2, 1)$ | $(0, 0)$ | $(1, 4)$  | $(3, -6)$ | $(2, 5)$  | $(-1, -4)$ |
| $E_2$ | $(-1, 2)$  | $(2, -1)$ | $(-2, 1)$ | $(0, 0)$ | $(-3, 0)$ | $(-1, 2)$ | $(-2, 1)$ | $(3, 0)$   |
| $C_1$ | $(0, -2)$  | $(-2, 0)$ | $(-4, 0)$ | $(0, 2)$ | $(-4, 2)$ | $(4, 0)$  | $(2, 0)$  | $(0, -2)$  |

Of course, we can also compute the image under  $f$  of the generators of the fundamental group of the genus 10 curves  $\hat{C}_3$  and  $\hat{C}_4$ .

### 3. PICARD NUMBER

**Lemma 6.** *Suppose  $D$  is a reduced (not necessarily irreducible) totally geodesic curve on a smooth complex two-ball quotient  $X$  with ordinary singularities only at  $k$  distinct points of respective multiplicities  $(b_1, \dots, b_k)$  and let us denote by  $D_i$  ( $i = 1, \dots, n$ ) its irreducible components,  $\hat{D}_i$  their normalization. Let  $\nu : \hat{D} = \cup_i \hat{D}_i \rightarrow D$  be the normalization of  $D$ . Then*

$$K_X \cdot D = 3 \sum_{i=1}^n (g(\hat{D}_i) - 1) \quad \text{and} \quad D \cdot D = \frac{1}{2} e(\hat{D}) + \tilde{\delta}, \quad \text{where} \quad \tilde{\delta} = \sum_{i=1}^k b_i(b_i - 1)$$

and  $e(\hat{D})$  is the Euler characteristic of  $\hat{D}$ .

*Proof.* Note that we are in the case of a (non necessarily connected) immersed smooth curve in a surface, with singularities given by intersections of transversal local branches. Moreover, it is well known that for a totally geodesic curve  $D$  in a two-ball quotient,  $c_1(K_{\hat{D}}) = \frac{2}{3} \nu^* c_1(K_X)$  (this is a simple and classical computation involving the curvature form on  $B_{\mathbb{C}}^2$ ). For instance, it follows from [CM, §1] that if  $M$  is a smooth compact  $m$ -ball quotient for some  $m$  and  $\omega_M$  is the Kähler form of the metric on  $M$  with constant holomorphic sectional

curvature  $-2$  then  $c_1(K_M) = (m+1)\omega_M$ . Now, if  $\nu : N \rightarrow M$  is a totally geodesic immersion of a smooth ball quotient  $N$  in  $M$ , then  $\nu^*\omega_M$  is the Kähler form of the metric on  $N$  with constant holomorphic sectional curvature  $-2$  hence the result). As a consequence, by the adjunction formula,

$$K_X \cdot D = \int_D c_1(K_X) = \frac{3}{2} \sum_i \int_{\widehat{D}_i} c_1(K_{\widehat{D}_i}) = 3 \sum_{i=1}^n (g(\widehat{D}_i) - 1).$$

Recall moreover from [BHPV, §II.11] that

$$g(D) = g(\widehat{D}) + \delta^{\text{an}}(D), \quad \text{where } g(\widehat{D}) = 1 + \sum_i (g(\widehat{D}_i) - 1) \text{ and } \delta^{\text{an}}(D) = \sum_{x \in D} \dim_{\mathbb{C}}(\nu_* \mathcal{O}_{\widehat{D}} / \mathcal{O}_D)$$

(here, the genus of a singular curve is its arithmetic genus). From the adjunction formula for embedded curves,  $2(g(D) - 1) = K_X \cdot D + D \cdot D$  and therefore,

$$D \cdot D = 2(g(D) - 1) - K_X \cdot D = 2(g(\widehat{D}) + \delta^{\text{an}}(D) - 1) - 3(g(\widehat{D}) - 1) = (1 - g(\widehat{D})) + 2\delta^{\text{an}}(D).$$

Finally, observe that in the case at hand,  $\delta^{\text{an}}(D) = \frac{1}{2} \sum_{i=1}^k b_i(b_i - 1) = \frac{1}{2} \widetilde{\delta}$ .  $\square$

**Lemma 7.** *With  $C$  either  $C_1$  or  $C_2$ , we have the following intersection matrix for the curves  $E_1, E_2, E_3$  and  $C$ :*

$$\begin{pmatrix} 5 & 13 & 9 & 11 \\ 13 & 5 & 9 & 7 \\ 9 & 9 & 9 & 9 \\ 11 & 7 & 9 & -1 \end{pmatrix}. \quad (7)$$

Moreover  $K_X \cdot E_i = 9 = K_X \cdot C$  for  $i = 1, 2, 3$ .

*Proof.* This follows immediately from Lemma 6 (here, all the involved curves are irreducible) and the results in §2.6 and §2.7. For example for  $E_3$ ,  $e(\widehat{E}_3) = -6$  and  $(b_1, b_2, b_3) = (1, 4, 1)$ , from which Lemma 6 implies that  $E_3 \cdot E_3 = 9$ .

As the normalization of each of the four curves is of genus 4, its intersection with  $K_X$  is 9 by Lemma 6. We leave the other computations to the reader and just observe that:

- a curve  $E_i$  can only intersect a curve  $E_j$  at  $\varpi^{-1}(P_2)$ ,
- two local branches of a curve  $C$  can only intersect at  $\varpi^{-1}(P_2)$ ,
- a curve  $C$  can only intersect a curve  $E_i$  at  $\varpi^{-1}(P_2)$  and  $\varpi^{-1}(P_3)$ .  $\square$

From now on, for any two divisors  $D$  and  $D'$  on  $X$ ,  $D \equiv D'$  will mean that  $D$  and  $D'$  are numerically equivalent.

**Lemma 8.**  *$E_1, E_2$  and  $C$  represent numerically linearly independent elements in the Néron-Severi group, where  $C = C_1$  or  $C_2$ .*

*Proof.* Assume that  $E_1, E_2$  and  $C$  satisfy numerically an identity  $aE_1 + bE_2 + cC \equiv 0$ . Since the minor  $\Delta$  of the above intersection matrix obtained by deleting the third row and column has determinant  $1296 \neq 0$ , we must have  $a = b = c = 0$ .  $\square$

**Corollary 1.** *The Picard number of  $X$  is 3.*

*Proof.* It follows from the previous lemma that the Picard number is at least 3, given by the classes of  $E_1, E_2$  and  $C$ . On the other hand,  $h^{1,1}(X) = 3$  by Lemma 3. Since the Picard number is bounded from above by  $h^{1,1}$ , we conclude that the Picard number is 3.  $\square$

**Proposition 9.** *The canonical line bundle  $K_X$  and  $E_3$  give rise to the same class in the Néron-Severi group. Moreover,  $K_X \equiv E_3 \equiv \frac{1}{2}E_1 + \frac{1}{2}E_2$ .*

*Proof.* From the discussions in the previous section, we know that  $E_1, E_2$  and  $C = C_1$  form a basis over  $\mathbb{Q}$  of the Néron-Severi group (which is torsion free, see Lemma 3).

Hence we may write  $K_X \equiv aE_1 + bE_2 + cC$  for some rational numbers  $a, b$  and  $c$ . By pairing with  $E_1, E_2$  and  $C$  respectively, we arrive at the system of equations

$$9 = 5a + 13b + 11c, \quad 9 = 13a + 5b + 7c \quad \text{and} \quad 9 = 11a + 7b - c,$$

so that  $(a, b, c)^t = \Delta^{-1}(9, 9, 9)^t = (\frac{1}{2}, \frac{1}{2}, 0)^t$ , where  $\Delta$  is the minor used in the proof of Lemma 8. Thus  $K_X \equiv \frac{1}{2}E_1 + \frac{1}{2}E_2$ . The same computation leads to  $E_3 \equiv \frac{1}{2}(E_1 + E_2)$  since  $E_3 \cdot E_i = K_X \cdot E_i$  for  $i = 1, 2$  and  $E_3 \cdot C = K_X \cdot C$ .  $\square$

We show in Remark 6 below that in fact  $E_3$  is linearly equivalent to  $K_X$  and  $E_1 + E_2$  is linearly equivalent to  $2K_X$ .

**Remark 2.** By the previous proposition, we also have  $K_X \equiv \frac{2}{3}(\frac{1}{2}E_1 + \frac{1}{2}E_2) + \frac{1}{3}E_3 = \frac{1}{3}(E_1 + E_2 + E_3)$ . This fact can be recovered directly from the description of  $X$  as an orbifold covering of  $R = \bar{\Gamma} \backslash B_{\mathbb{C}}^2$  as in §2.

We use the notation of §1.2. Let  $q : \bar{Q} = \mathbb{P}_{\mathbb{C}}^2 \rightarrow R = \mathbb{P}_{\mathbb{C}}^2 / \Sigma_3$  be the projection. First, we compute the canonical divisor  $K_R$  of  $R$ . We have  $K_R = aD_A = 2aD_B$  for some  $a \in \mathbb{Q}$  (see [DM2, §11.4 and Proposition 11.5] for a description of  $\text{Pic}(R)$ ). If we denote by  $L = \mathcal{O}(1)$  the positive generator of  $\text{Pic}(\mathbb{P}_{\mathbb{C}}^2)$ , we have  $-3L = K_{\mathbb{P}_{\mathbb{C}}^2} = q^*K_R + 3L = 6aL + 3L$  as  $q$  branches at order 2 along  $D_A$ , and  $D_A$  has three lines as a preimage in  $\mathbb{P}_{\mathbb{C}}^2$ . Hence  $K_R = -D_A = -2D_B$ .

Now, the orbifold canonical divisor of  $\bar{\Gamma} \backslash B_{\mathbb{C}}^2$  is  $K_R + \frac{3-1}{3}D_A + \frac{4-1}{4}D_B = (-1 + \frac{2}{3} + \frac{3}{8})D_A = \frac{1}{24}D_A = \frac{1}{12}D_B$ . In particular, as  $\varpi^*D_B = 4(E_1 + E_2 + E_3)$ , we get the result.

**Remark 3.** Proceeding as above, one can also find the following relations

$$4E_3 \equiv C_1 + C_3, \quad 4E_3 \equiv C_2 + C_4 \quad \text{and} \quad 3E_3 \equiv E_1 + C_1 + C_2.$$

#### 4. GEOMETRY OF A GENERIC FIBER OF THE ALBANESE MAP

Let  $\alpha : X \rightarrow T$  be the Albanese map of  $X$ . From  $\Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$ , we know that  $T$  is an elliptic curve, and in particular,  $\alpha$  is onto. Moreover, note that since the image of  $\alpha$  is a curve, the fibers of  $\alpha$  are connected (see [U, Proposition 9.19]). Let  $D$  be a curve on  $X$ . The mapping  $\alpha$  induces a mapping  $\alpha|_D : D \rightarrow T$ . Suppose  $F$  is the generic fiber of  $\alpha$ . Then the degree of  $\alpha|_D$  is given by  $D \cdot F$ .

**Lemma 9.** Let  $m, n, p$  be the degrees of  $E_1, E_2$ , and  $C = C_1$ , respectively, over the Albanese torus  $T$  of  $X$ . The generic fiber  $F$  of the Albanese fibration of  $X$  satisfies

$$F \equiv \frac{1}{72}((-3m + 5n + 2p)E_1 + (5m - 7n + 6p)E_2 + 2(m + 3n - 4p)C).$$

*Proof.* As in the proof of Proposition 9, we can write  $F \equiv aE_1 + bE_2 + cC$  for some rational numbers  $a, b, c$ , and using the same minor  $\Delta$  used in the proof of Lemma 8, we get  $(a, b, c)^t = \Delta^{-1}(m, n, p)^t$ .  $\square$

**Lemma 10.** The degrees of  $E_1, E_2, C = C_1$  over the Albanese torus  $T$  of  $X$  are given by

$$m = 60, \quad n = 12, \quad p = 24.$$

*Proof.* Let  $D$  represent one of the curves  $E_1, E_2, C$ ,  $\nu : \hat{D} \rightarrow D$  the normalization of  $D$  and  $\hat{\alpha} = \alpha \circ \nu$ . Let  $\omega$  be a positive  $(1, 1)$  form on  $T$ . Then the degree of  $D$  over  $T$  is given by  $\deg(D) = \frac{\int_D \alpha^* \omega}{\int_T \omega}$ . The key is to find the degree from the information of the explicit curves that we have. For this purpose, we use an analogue of the Riemann bilinear relations. Let

$\eta$  be a holomorphic 1-form on the smooth Riemann surface  $\widehat{D}$ . Let  $\{u_i, v_i\}$  be a basis of  $\pi_1(\widehat{D})$  as studied in §2.8. Then the Riemann bilinear relation (cf. [GH, p. 231]) states that

$$\int_{\widehat{D}} \sqrt{-1} \eta \wedge \bar{\eta} = \sqrt{-1} \sum_{i=1}^4 \left[ \int_{u_i} \overline{\eta} \int_{v_i} \eta - \int_{v_i} \overline{\eta} \int_{u_i} \eta \right]$$

where we use the same notation for an element of  $\pi_1(\widehat{D})$  and its image in  $H_1(\widehat{D}, \mathbb{Z})$ . Let us write  $T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$  where  $\text{Im } \tau > 0$ . Let  $\omega_T = \sqrt{-1} dz \wedge \bar{dz}$  be the standard  $(1, 1)$  form on  $\mathbb{C}$  and hence  $T$ . The above formula gives

$$\int_T \omega_T = \sqrt{-1}(\bar{\tau} - \tau). \quad (8)$$

Pulling back to  $D$ , the above formula gives

$$\begin{aligned} \int_D \alpha^* \omega_T &= \int_{\widehat{D}} \hat{\alpha}^* \omega_T = \int_{\widehat{D}} \sqrt{-1} \hat{\alpha}^* dz \wedge \hat{\alpha}^* \bar{dz} \\ &= \sqrt{-1} \sum_{i=1}^4 \left[ \int_{u_i} \hat{\alpha}^* dz \int_{v_i} \overline{\hat{\alpha}^* dz} - \int_{v_i} \hat{\alpha}^* dz \int_{u_i} \overline{\hat{\alpha}^* dz} \right] \\ &= \sqrt{-1} \sum_{i=1}^4 \left[ \int_{\hat{\alpha}_*(u_i)} dz \int_{\hat{\alpha}_*(v_i)} \overline{dz} - \int_{\hat{\alpha}_*(v_i)} dz \int_{\hat{\alpha}_*(u_i)} \overline{dz} \right]. \end{aligned} \quad (9)$$

In the above,  $\hat{\alpha}_* : H_1(\widehat{D}, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z}) \cong H_1(X, \mathbb{Z}) \cong \mathbb{Z}^2$  refers to the map on 1-cycles induced by  $\hat{\alpha}$ . Hence the right-hand side of the above expression in terms of the notation in §2.8 is (up to sign)

$$\sqrt{-1} \sum_{i=1}^4 \left[ \int_{f(u_i)} dz \int_{f(v_i)} \overline{dz} - \int_{f(v_i)} dz \int_{f(u_i)} \overline{dz} \right] = \left[ \sum_{i=1}^4 \det(f(u_i), f(v_i)) \right] \sqrt{-1}(\bar{\tau} - \tau), \quad (10)$$

where  $\det(f(u_i), f(v_i))$  stands for the determinant of the two by two matrix formed by the two vectors  $f(u_i)$  and  $f(v_i)$  from the table in §2.8. Notice that the resulting number will be positive if and only if the orientation on  $\widehat{D}$  coming from the choice of  $(u_1, v_1, \dots, u_4, v_4)$  as a symplectic basis of  $H_1(\widehat{D}, \mathbb{Z})$ , and the orientation on  $T$  induced by the choice of the basis  $(e_1, e_2)$  of  $H_1(T, \mathbb{Z})$  are compatible (i.e. both are the same, or the opposite, as the one induced by the respective complex structures).

Substituting into (9) and (10) the values of  $f(u_i)$  and  $f(v_i)$  from the table in §2.8, we conclude the values of  $-60, -12, -24$  for the values of  $\sum_{i=1}^4 \det(f(u_i), f(v_i))$  in the case of  $E_1, E_2$  and  $C$  respectively. We conclude from (8), (9) and (10) that the degrees  $m, n, p$  are given by 60, 12 and 24 respectively (and that the orientation on  $\widehat{D}$  and  $T$  are not compatible).  $\square$

**Theorem 3.** *A fiber of the Albanese map  $\alpha : X \rightarrow T$  represents the same numerical class as  $-E_1 + 5E_2$ , and the genus of a generic fiber  $F$  is 19.*

*Proof.* Substituting the values of  $m, n, p$  from the previous lemma into Lemma 9, we conclude that  $F$  represents the same class as  $-E_1 + 5E_2$  in the Néron-Severi group. Hence  $F \cdot K_X = -E_1 \cdot K_X + 5E_2 \cdot K_X = 36$ , so that  $g = 19$  follows from the adjunction formula  $2(g - 1) = (K_X + F) \cdot F = K_X \cdot F$ .  $\square$

## 5. GEOMETRY OF THE ALBANESE FIBRATION

**5.1.** Let  $X_s$  be the fiber of the Albanese fibration  $\alpha$  at  $s \in T$ . It is connected (see §4). Now  $g(X_s) \geq 2$ , because  $X$  has negative holomorphic sectional curvature. Although we will not need this in the sequel, we observe that the fibration cannot be locally holomorphically trivial. Otherwise there is a smooth non-trivial family of holomorphic mappings from  $X_s$  (where  $s \in T$  is generic) to  $X$ . However, a holomorphic map is harmonic with respect to

any Kähler metric on  $X_s$  and the Poincaré metric on  $X$ . As the Poincaré metric on  $X$  is strictly negative, it follows from uniqueness of harmonic maps to a negatively curved Kähler manifold in its homotopy class that the family is actually a singleton, a contradiction.

**5.2.** The result below is a refinement of Lemma VI.5 in [Be]. As usual, if  $D$  is a (not necessarily reduced) curve, we denote by  $g(D)$  its arithmetic genus (see [BHPV, §II.11]).

**Proposition 10.** *Let  $X$  (resp.  $C$ ) be a smooth complex surface (resp. curve) and  $\pi : X \rightarrow C$  a surjective morphism with connected fibers. Let  $D = \sum_{i=1}^k m_i D_i$ , ( $m_i \geq 1$ ) be a singular fiber of  $\pi$  and let  $D^{\text{red}} = \sum_{i=1}^k D_i$  be the reduced divisor associated to  $D$ . Let  $\nu : \widehat{D^{\text{red}}} \rightarrow D^{\text{red}}$  be the normalization. For any  $x$  in the support of  $D^{\text{red}}$ , we define  $\delta_x^{\text{top}} := \dim_{\mathbb{C}}(\nu_* \mathbb{C}_{\widehat{D^{\text{red}}}} / \mathbb{C}_{D^{\text{red}}}) = \sharp \nu^{-1}(x) - 1$  the number of (local) irreducible components of  $D^{\text{red}}$  at  $x$  minus 1 and  $\delta_x^{\text{an}} := \dim_{\mathbb{C}}(\nu_* \mathcal{O}_{\widehat{D^{\text{red}}}} / \mathcal{O}_{D^{\text{red}}})$  so that  $\mu_x := 2\delta_x^{\text{an}} - \delta_x^{\text{top}}$  is the Milnor number of  $D^{\text{red}}$  at  $x$ . We also set  $\mu = \sum_{x \in D^{\text{red}}} \mu_x$ . Then, we have*

$$e(D^{\text{red}}) - e(X_s) = \mu + \sum_{i=1}^k (m_i - 1) K_X \cdot D_i - (D^{\text{red}})^2 \quad (11)$$

where  $s$  is a generic point of  $C$ .

*Proof.* From the proof of Lemma VI.5 in [Be], we immediately get

$$e(D^{\text{red}}) = \mu + 2\chi(\mathcal{O}_{D^{\text{red}}}) = \mu + e(X_s) + 2(\chi(\mathcal{O}_{D^{\text{red}}}) - \chi(\mathcal{O}_D)),$$

where we used the fact that the arithmetic genus of the fibers of a morphism from a surface onto a curve is constant and  $e(X_s) = -(K_X + D) \cdot D = 2\chi(\mathcal{O}_D)$ . Now, since  $D^2 = 0$ ,

$$\begin{aligned} 2(\chi(\mathcal{O}_{D^{\text{red}}}) - \chi(\mathcal{O}_D)) &= (K_X + D) \cdot D - (K_X + D^{\text{red}}) \cdot D^{\text{red}} \\ &= K_X \cdot (D - D^{\text{red}}) - (D^{\text{red}})^2 \\ &= \sum_{i=1}^k (m_i - 1) K_X \cdot D_i - (D^{\text{red}})^2. \end{aligned}$$

That  $2\delta_x^{\text{an}} - \delta_x^{\text{top}}$  is the Milnor number of  $D^{\text{red}}$  at  $x$  is proved in [BG, Proposition 1.2.1].  $\square$

**Remark 4.** Recall that Milnor numbers are nonnegative. In the notation of Proposition 10,  $\mu_x = 0$  if and only if  $D^{\text{red}}$  is smooth at  $x$  and if  $\mu_x = 1$  it is easily seen that the singularity of  $D^{\text{red}}$  at  $x$  is nodal (see Lemmas 1.2.1 and 1.2.4 in [BG] for instance).

**Corollary 2.** *Let  $I \subset T$  be the set of singular values of the Albanese fibration  $\alpha$ . Then*

- (a)  $\sum_{s_o \in I} (e(X_{s_o}) - e(X_s)) = 3$  where  $X_s$  is a generic fiber,
- (b) the cardinality of  $I$  is at most 3,
- (c) the fibers of  $\alpha$  are reduced, and therefore  $X_{s_o}$  is singular for at least one  $s_o \in I$ ,
- (d) the total number of singular points in the fibers is at most 3 and if equality holds, the three singularities are nodal and the fibration is stable. More precisely,

$$\sum_{s_o \in I} \left( \sum_{x \in X_{s_o}} \mu_x \right) = 3. \quad (12)$$

*Proof.* Note first that there are no rational or elliptic curves in  $X$  since the holomorphic sectional curvature of a ball quotient is negative.

(a) From the standard formula for the Euler number of a holomorphic fibration (see [Be, Lemma VI.4] or [BHPV, Proposition III.11.4]), we have

$$3 = e(X) = e(T) \cdot e(X_s) + \sum_{s_o \in I} n_{s_o} = \sum_{s_o \in I} n_{s_o},$$

where  $n_{s_o} = e(X_{s_o}) - e(X_s)$  for  $s \in T_o := T - I$ . Here we used the fact that the Euler characteristic of  $T$  vanishes.

(b) It is well known (see [BHPV, Remark III.11.5]), and it can be easily recovered from Proposition 10, that  $n_{s_o} \geq 0$  with equality if and only if  $X_{s_o}$  is a multiple fiber with  $(X_{s_o})^{\text{red}}$  smooth elliptic. But as we noticed above, this is impossible in our case thus  $n_{s_o} > 0$  for any  $s_o \in I$ . Since  $\sum_{s_o \in I} n_{s_o} = 3$ , we conclude in particular that  $|I| \leq 3$  (and each  $n_{s_o} \leq 3$ ).

(c) Let  $D = \sum_{i=1}^k m_i D_i$  be a fiber of  $\alpha$ , where the  $D_i$ 's are the irreducible components of  $D$ , i.e.  $m_i \geq 1$  for all  $i$  and assume that  $m_1 \geq 2$ . If we denote by  $\hat{D}_i$  the normalization of  $D_i$ , then

$$K_X \cdot D_i \geq (K_X + D_i) \cdot D_i = 2(g(D_i) - 1) \geq 2(g(\hat{D}_i) - 1) \geq 2$$

where the last inequality holds because  $X$  contains no curve of geometric genus 0 or 1. Recall that by Zariski's lemma (see [BHPV, Lemma III.8.2]) the self-intersection of any effective cycle supported on  $D^{\text{red}}$  must be nonpositive, and it is equal to zero if and only if it is proportional to  $D$ . Then, by (a) and formula (11),

$$3 \geq e(D^{\text{red}}) - e(X_s) \geq \mu + \sum_{i=1}^k (m_i - 1) K_X \cdot D_i - (D^{\text{red}})^2$$

which is only possible if  $m_1 = 2$ ,  $g(D_1) = 2$  and  $m_i = 1$  for  $i \geq 2$ .

If  $k \geq 2$  then  $\mu \geq 1$  and  $(D^{\text{red}})^2 < 0$  by Zariski's lemma so that we get a contradiction. If  $k = 1$  then by Theorem 3,  $18 = g(D) - 1 = 2(g(D_1) - 1) = 2$ , which also leads to a contradiction.

(d) is a consequence of the previous points, equation (11) and Remark 4.  $\square$

**5.3.** Recall from §1.4 that the normalizer  $N$  of  $\Pi$  in  $\bar{\Gamma}$  is generated by the element  $j^4$  of order 3 and  $\Pi$ , and the automorphism group  $\Sigma$  of  $X$  is given by the group  $N/\Pi$ , which has order 3. Denote by  $\sigma$  the automorphisms of  $B_{\mathbb{C}}^2$  and of  $X$  induced by  $j^4$ . If  $\xi = (z_1, z_2) \in B_{\mathbb{C}}^2$ , then  $\sigma(\xi) = (\omega z_1, \omega z_2)$  where  $\omega = \zeta^4$  is a non trivial cube root of unity.

The Albanese map  $\alpha : X \rightarrow T = \mathbb{C}/\Lambda$  can be lifted to a holomorphic map  $\alpha_0 : B_{\mathbb{C}}^2 \rightarrow \mathbb{C}$  so that  $\alpha_0(O) = 0$  (choosing  $\Pi O \in X$  as base point when defining  $\alpha$ ):

$$\begin{array}{ccc} & & \mathbb{C} \\ & \nearrow \alpha_0 & \downarrow \\ B_{\mathbb{C}}^2 & \xrightarrow{\quad} & X \xrightarrow{\quad} T \end{array}$$

If  $\pi \in \Pi$ , then  $\alpha_0(\pi\xi) - \alpha_0(\xi) \in \Lambda$  is independent of  $\xi \in B_{\mathbb{C}}^2$ , and so there is a map  $\theta_0 : \Pi \rightarrow \Lambda$  such that  $\alpha_0(\pi\xi) = \alpha_0(\xi) + \theta_0(\pi)$  for all  $\xi \in B_{\mathbb{C}}^2$  and  $\pi \in \Pi$ . Since  $\theta_0$  is a homomorphism, it factors through our abelianization map  $f : \Pi \rightarrow \mathbb{Z}^2$ , see §1.4. So there is a homomorphism  $\theta : \mathbb{Z}^2 \rightarrow \Lambda$  such that

$$\alpha_0(\pi\xi) = \alpha_0(\xi) + \theta(f(\pi)) \quad \text{for all } \xi \in B_{\mathbb{C}}^2 \text{ and } \pi \in \Pi. \quad (13)$$

By the universal property of the Albanese map, there is an automorphism  $\sigma_T : T \rightarrow T$  such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & T \\ \downarrow \sigma & & \downarrow \sigma_T \\ X & \xrightarrow{\alpha} & T \end{array} \quad (14)$$

If the automorphism is trivial, then  $\alpha_0(\sigma(\xi)) - \alpha_0(\xi) \in \Lambda$  for all  $\xi \in B_{\mathbb{C}}^2$ , and so is constant. Since  $\sigma(O) = O$ ,  $\alpha_0(j^4\xi) = \alpha_0(\xi)$  for all  $\xi$ , and this implies that  $\theta(f(j^4\pi j^{-4})) = \theta(f(\pi))$  for all  $\pi \in \Pi$ . But then (3) implies that  $\theta = 0$ , because  $I - \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$  is non-singular hence  $\Pi\xi \mapsto \alpha_0(\xi)$  is a holomorphic function  $X \rightarrow \mathbb{C}$ , and so is constant because  $X$  is compact, contradicting surjectivity of  $\alpha$ .

As a consequence,  $\Sigma$  acts non trivially on  $T$  and since  $\sigma(O) = O$ , the action of  $\Sigma$  fixes the point  $\alpha(\Pi O) = 0 + \Lambda$ . From this and  $|\Sigma| = 3$ , it follows immediately that the elliptic curve has to be  $T = \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ , and the vertical map  $\sigma_T$  on the right in (14) is  $z + \Lambda \mapsto \omega^t z + \Lambda$  with  $t = 1$  or  $2$ . Indeed, the automorphism  $\sigma_T$  which fixes  $0 + \Lambda$  is induced by a nontrivial  $\mathbb{C}$ -linear automorphism of  $\mathbb{C}$  preserving  $\Lambda$  (see [Be, Proposition V.12] for instance). Since it has order 3, it must be multiplication by  $\omega^t$ , where  $t = 1$  or  $2$ . Hence  $\Lambda$  contains 1 and  $\omega$  (after renormalization of the lattice).

It follows that there are precisely 3 fixed points of  $\Sigma$  on  $T$ : a fundamental domain of  $T$  consists of two equilateral triangles and the fixed points are given by a vertex and the centroid of each of the two triangles i.e. are the points  $p_j = j(2 + \omega)/3 + \Lambda$ ,  $j = 0, 1, -1$  (notice that  $(1 - \omega)^{-1} = (2 + \omega)/3$ ). In particular, we have proved

**Lemma 11.** *The action of  $\Sigma$  descends to a non-trivial action of  $T$ . There are three fixed points in the action of  $\Sigma$  on  $T$ . The elliptic curve  $T$  is isomorphic to  $\mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ .*

**5.4.** Our purpose now is to determine the fixed points of  $\text{Aut}(X)$ . Let  $p_j = j(2 + \omega)/3 + \Lambda$ ,  $j = 0, 1, -1$ , be the fixed points of  $\Sigma$  on  $T$ , as given by Lemma 11.

- Proposition 11.** (a) *There are altogether nine fixed points of  $\text{Aut}(X)$  on  $X$ .*  
 (b) *The points  $O_1, O_2$  and  $O_3$  mentioned in §2 are fixed points of  $\Sigma$ , all lie in the same fiber  $\alpha^{-1}(p_0)$ .*  
 (c) *The other fixed points are 6 of the 288 points lying in  $\varpi^{-1}(P_5)$  (see §2).*  
 (d) *Each of the fibers  $\alpha^{-1}(p_j)$  for  $j = 1, 0, -1$  contains exactly three of the nine fixed points of  $\text{Aut}(X)$ .*  
 (e) *The fixed points  $O_i$ ,  $i = 1, 2, 3$  are of type  $\frac{1}{3}(1, 1)$ , and the other six fixed points are of type  $\frac{1}{3}(1, 2)$ .*

*Proof.* (a) and (c): If  $\Pi\xi$  is fixed by  $\sigma$ , then  $\Pi j^4 \xi = \Pi\xi$ , and so  $\pi j^4 \xi = \xi$  for some  $\pi \in \Pi$ . This implies that  $\pi j^4$  has finite order. It cannot be trivial, since  $\Pi$  is torsion free. So there is an element  $t$ , in the list of representative nontrivial elements of finite order in  $\bar{\Gamma}$  given in Proposition 1, or the inverse of one of these, such that  $\pi j^4 = gtg^{-1}$  for some  $g \in \bar{\Gamma}$ . Thus  $gtg^{-1}j^{-4} \in \Pi$ . Since the elements  $b^\mu k$ ,  $\mu = 0, 1, -1$  and  $k \in K$ , form a set of coset representatives for  $\Pi$  in  $\bar{\Gamma}$ , and since  $j^4$  normalizes  $\Pi$ , we can assume that  $g = b^\mu k$  for some  $\mu$  and  $k$ .

So we search through the finite set of elements  $b^\mu k t k^{-1} b^{-\mu} j^{-4}$ , checking which are in  $\Pi$ . We find that  $b^\mu k t k^{-1} b^{-\mu} j^{-4} \in \Pi$  only happens for  $t = j^4$  and  $t = buv$ . When  $t = j^4$ , we have  $b^\mu k t k^{-1} b^{-\mu} j^{-4} = b^\mu j^4 b^{-\mu} j^{-4}$ , independent of  $k$ . We find that these three elements are in  $\Pi$ , i.e.,  $b^\mu j^4 b^{-\mu} j^{-4} = \pi_\mu$  for some  $\pi_\mu \in \Pi$ . Explicitly,

$$b^\mu j^4 b^{-\mu} j^{-4} = \pi_\mu \in \Pi, \text{ for } \pi_0 = 1, \pi_1 = a_2 a_1^{-2} a_3^{-3} a_1^{-1} \text{ and } \pi_{-1} = a_2^2 a_1 a_3 a_1^{-1}. \quad (15)$$

Thus  $\pi_\mu j^4 (b^\mu O) = b^\mu (j^4(O)) = b^\mu O$ , so that the three points  $\Pi(b^\mu O)$  are fixed by  $\sigma$ .

For  $t = buv$ , we find that  $b^\mu k t k^{-1} b^{-\mu} j^{-4} \in \Pi$  for only 18 pairs  $(\mu, k)$ . This means that  $\sigma$  fixes  $\Pi(b^\mu k.Q)$  for these 18  $(\mu, k)$ 's, where  $Q$  is the point defined in §1.3, equation (2). If  $(\mu, k)$  satisfies  $b^\mu k t k^{-1} b^{-\mu} j^{-4} \in \Pi$ , then so does  $(\mu, k j^4)$ , since we can write  $b^\mu j^4 = \pi_\mu j^4 b^\mu$  for some  $\pi_\mu \in \Pi$ , as we have just seen. Moreover,  $\Pi(b^\mu k j^4.Q) = \Pi(b^\mu k.Q)$ , since  $k j^4 = j^4 k$  and so

$$\Pi(b^\mu k j^4.Q) = \Pi(\pi_\mu j^4 b^\mu k.Q) = \Pi(j^4 b^\mu k.Q) = \sigma(\Pi(b^\mu k.Q)) = \Pi(b^\mu k.Q).$$

So we need only consider six of the  $(\mu, k)$ 's, and correspondingly setting

$$\begin{aligned} h_1 &= b^{-1} u v j^3, & h_2 &= u^{-1} v j, & h_3 &= b u v^2 j^2, \\ h_4 &= b^{-1} v^2 u j^3, & h_5 &= v j^2, & h_6 &= b v u^{-1} v, \end{aligned}$$

we have  $h_i(buv)h_i^{-1}j^{-4} = \pi'_i \in \Pi$  for  $i = 1, \dots, 6$ ; explicitly,

$$\begin{aligned}\pi'_1 &= a_2^2 a_1 a_3^3, & \pi'_2 &= j^8 a_1 j^4, & \pi'_3 &= j^8 a_1 a_2^3 j^4 a_2 a_1 a_2^{-2} a_1^{-1}. \\ \pi'_4 &= a_3^3 a_1^2 a_3^3, & \pi'_5 &= j^4 a_1^{-1} a_2^{-1} j^8, & \pi'_6 &= a_2 a_1^{-1}.\end{aligned}$$

These six points  $\Pi h_i Q$  are distinct, as we see by checking that  $(b^{\mu'} k')(buv)^{\epsilon}(b^{\mu} k)^{-1}$  is not in  $\Pi$  for  $\epsilon = 0, 1, 2$  when  $(\mu', k')$  and  $(\mu, k)$  are distinct pairs in the above list.

Note that (a) corresponds to the case of Proposition 1.2 (2)(b) in Keum [K], the latter following from Lefschetz fixed point formula and holomorphic Lefschetz fixed point formula.

(b) and (d): It is evident that if  $x \in X$  is fixed by  $\sigma$  then  $\alpha(x) \in T$  is fixed by  $\Sigma$  and so is a  $p_j$ . Writing  $\alpha(\Pi\xi) = \alpha_0(\xi) + \Lambda$ , as before, where  $\alpha_0(O) = 0$ , we proved in §5.3 that  $\alpha_0(j^4\xi) = \omega^t \alpha_0(\xi)$  for all  $\xi \in B_{\mathbb{C}}^2$  for  $t = 1$  or  $2$ , and so  $\theta(f(j^4\pi j^{-4})) = \omega^t \theta(f(\pi))$ . We may assume that  $\theta(m, n) = m - n\omega^t$ , for writing  $\theta(m, n) = (rm + sn) + (r'm + s'n)\omega$ , the last condition allows us to express  $r'$  and  $s'$  in terms of  $r$  and  $s$ , and we find that  $\theta(m, n) = (r + s\omega)(m - n\omega)$  when  $t = 1$ , and that  $\theta(m, n) = (-\omega)(s + r\omega)(m - n\omega^2)$  when  $t = 2$ . We assume that  $t = 1$ , since the case  $t = 2$  may be dealt with similarly. As we saw above,  $\theta \neq 0$ , and so  $r + s\omega \neq 0$ , and since  $\theta(f(\pi)) \in (r + s\omega)\Lambda$  for all  $\pi \in \Pi$ , the map  $\tilde{\alpha} : \Pi\xi \mapsto \alpha_0(\xi)/(r + s\omega) + \Lambda$  is well-defined  $X \rightarrow T$ . So  $\tilde{\alpha} = g \circ \alpha$  for some holomorphic  $g : T \rightarrow T$ . Let  $\tilde{g} : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic lifting of  $g$ . It is easy to see that  $\tilde{g}(z) = z/(r + s\omega) + \lambda_0$  for a fixed  $\lambda_0 \in \Lambda$ . Since  $g$  is well-defined, we must have  $1/(r + s\omega) \in \Lambda$ . So  $r + s\omega = (-\omega)^j$  for some  $j \in \{0, \dots, 5\}$ , and therefore  $g$  is an automorphism of  $T$ . We may therefore replace  $\alpha$  by  $\tilde{\alpha}$ , the  $\theta$  of which is  $(m, n) \mapsto m - n\omega$ .

Now  $b j^4 b^{-1} = \pi_1 j^4$  for  $\pi_1$  as above, and  $f(\pi_1) = (-2, -5)$ , so

$$\alpha_0(b j^4 b^{-1} \xi) = \alpha_0(\pi_1 j^4 \xi) = \alpha_0(j^4 \xi) - 2 + 5\omega = \omega \alpha_0(\xi) - 2 + 5\omega.$$

In particular, taking  $\xi = bO$ , we have  $\alpha_0(bO) = \omega \alpha_0(bO) - 2 + 5\omega$ , so that

$$\alpha_0(bO) = \frac{2 + \omega}{3}(-2 + 5\omega) = \omega - 3 \in \Lambda.$$

Hence  $\alpha(\Pi b.O) = \alpha(\Pi O) = p_0$ .

Similarly,  $b^{-1} j^4 b = \pi_{-1} j^4$  for  $\pi_{-1}$  as above, and  $f(\pi_{-1}) = (-5, 1)$ , so that

$$\alpha_0(b^{-1} j^4 b \xi) = \alpha_0(\pi_{-1} j^4 \xi) = \alpha_0(j^4 \xi) + \theta(f(\pi_{-1})) = \omega \alpha_0(\xi) - 5 - \omega.$$

So taking  $\xi = b^{-1}O$ , we have  $\alpha_0(b^{-1}O) = \omega \alpha_0(b^{-1}O) - 5 - \omega$ , so that

$$\alpha_0(b^{-1}O) = \frac{2 + \omega}{3}(-5 - \omega) = -3 - 2\omega \in \Lambda.$$

Hence  $\alpha(\Pi b^{-1}.O) = \alpha(\Pi O) = p_0$  too. So (b) is proved.

Recall now that  $h_i(buv)h_i^{-1}j^{-4} = \pi'_i \in \Pi$  for  $i = 1, \dots, 6$ , and so

$$\alpha_0(h_i(buv)h_i^{-1}\xi) = \alpha_0(\pi'_i j^4 \xi) = \alpha_0(j^4 \xi) + \theta(f(\pi'_i)) = \omega \alpha_0(\xi) + \theta(f(\pi'_i)).$$

In particular, taking  $\xi = h_i Q$ , we get  $\alpha_0(h_i Q) = \omega \alpha_0(h_i Q) + \theta(f(\pi'_i))$ , so that

$$\alpha_0(h_i Q) = \frac{2 + \omega}{3} \theta(f(\pi'_i)).$$

Calculating

$$\begin{aligned}f(\pi'_1) &= (-6, 2), & f(\pi'_2) &= (-4, 1), & f(\pi'_3) &= (1, -6), \\ f(\pi'_4) &= (-4, 0), & f(\pi'_5) &= (-4, 3), & f(\pi'_6) &= (-3, -2),\end{aligned}$$

we have

$$\begin{aligned}\theta(f(\pi'_1)) &= -6 - 2\omega \equiv 1 - (1 - \omega) & \theta(f(\pi'_4)) &= -4 & \equiv -1 \\ \theta(f(\pi'_2)) &= -4 - \omega \equiv 1 + (1 - \omega) & \text{and } \theta(f(\pi'_5)) &= -4 - 3\omega \equiv -1 \\ \theta(f(\pi'_3)) &= 1 + 6\omega \equiv 1 & \theta(f(\pi'_6)) &= -3 + 2\omega \equiv -1 + (1 - \omega),\end{aligned}$$

where the congruences are modulo 3. Hence  $\alpha(\Pi h_i Q) = \frac{2 + \omega}{3} + \Lambda$  for  $i = 1, 2, 3$  and  $\alpha(\Pi h_i Q) = -\frac{2 + \omega}{3} + \Lambda$  for  $i = 4, 5, 6$ . So (d) is proved.

(e) If  $\gamma \in \bar{\Gamma}$ , then writing  $\gamma.(z_1, z_2) = (w_1, w_2)$ , a routine calculation shows that

$$\begin{pmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_2}{\partial z_1} \\ \frac{\partial w_1}{\partial z_2} & \frac{\partial w_2}{\partial z_2} \end{pmatrix},$$

evaluated at  $\xi = (z_1, z_2)$ , equals

$$\frac{\zeta^2/(r-1)}{(\gamma_{31}\kappa z_1 + \gamma_{32}\kappa z_2 + \gamma_{33})^2} \begin{pmatrix} \kappa z_2 \bar{\gamma}_{23} + (r-1) \bar{\gamma}_{22} & -(\kappa z_2 \bar{\gamma}_{13} + (r-1) \bar{\gamma}_{12}) \\ -(\kappa z_1 \bar{\gamma}_{23} + (r-1) \bar{\gamma}_{21}) & \kappa z_1 \bar{\gamma}_{13} + (r-1) \bar{\gamma}_{11} \end{pmatrix},$$

where  $\kappa = \sqrt{r-1}$ . Taking  $\gamma = h_i(buv)h_i^{-1}$  and  $\xi = Q = (c_1/\kappa, c_2/\kappa)$  as given in (2), we find that this matrix has eigenvalues  $\omega^{\pm 1}$ . If instead we take  $\gamma = b^\mu j^4 b^{-\mu}$ , and  $\xi = b^\mu O$ , for  $\mu = 0, 1, -1$ , we find that the matrix is  $\omega I$ .

Note that (e) is also stated as one of the cases in [K, Proposition 1.2], and was observed by Igor Dolgachev as well.  $\square$

**Remark 5.** *We do not know whether the fibration  $\alpha$  is semistable. With some more effort, we can show that the fiber  $\alpha^{-1}(p_0)$  is smooth at each of the three points  $O_i$  and that  $\alpha$  is not semistable if and only if the only singularity of  $\alpha$  is a tacnode at one of the six other fixed points (see [CKY, Proposition 5]).*

**Remark 6.** *We can now show that  $E_3$  is linearly equivalent to  $K_X$  and  $E_1 + E_2$  is linearly equivalent to  $2K_X$  as follows. Consider the singular quotient surface  $S = X/\Sigma$ . It is simply connected. For recall from §1.4 that the normalizer  $N$  of  $\Pi$  in  $\bar{\Gamma}$  is generated by  $j^4$ , which has order 3, and by the elements  $a_1, a_2$  and  $a_3$ . In the proof of Proposition 11, we saw that  $j^4 a_1 j^8 = \pi'_2 j^4$  and  $a_2 a_1^{-1} j^4 = \pi'_6 j^4$  are conjugates of  $bu v$ , and so are torsion elements of  $N$ . Hence  $a_1$  and  $a_2$  can be written as words in torsion elements of  $N$ . Using the expression of  $j^4 a_3 j^{-4}$  in §1.4, we see that  $a_3$  can also be so written. Thus  $N$  is generated by elements having fixed points, and we can apply the main result of [A].*

Now, as  $E_3$  is globally fixed by  $\Sigma$ , it descends to a Weil divisor  $D_3$  on  $S$ . If  $K_S$  denotes the canonical divisor on  $S$  (which is only a  $\mathbb{Q}$ -Cartier divisor), then the pullback of  $K_S$  by the quotient map is  $K_X$  and it follows from Proposition 9 that  $D_3$  and  $K_S$  are numerically equivalent on  $S$ . However, since  $S$  is simply connected, this implies by the Universal Coefficient Theorem that  $D_3$  and  $K_S$  are linearly equivalent hence  $E_3$  and  $K_X$  are linearly equivalent. The same reasoning applies to  $E_1 + E_2$  and  $2K_X$ . Finally, note that  $S$  is a surface of general type, as  $K_S$  is effective and  $K_S^2 = 3$ .

**5.5.** Ngaiming Mok has kindly drawn to our attention the following problem which was open and of interest in the geometric study of complex ball quotients.

**Question 1.** *Does there exist a homomorphism  $f : X \rightarrow R$  from a smooth complex ball quotient  $X$  to a Riemann surface  $R$  with a non-totally geodesic singular fiber?*

There are very few explicit examples of mappings from a complex ball quotient to a Riemann surface. The known ones described by Deligne-Mostow, Mostow, Livné, Toledo and Deraux all have totally geodesic singular fibers, cf. [DM2], [T] or [Der2] and the references therein. We now show that the surface studied in this note provides such an example.

**Theorem 4.** *No singular fiber of the Albanese fibration  $\alpha : X \rightarrow T$  is totally geodesic.*

*Proof.* Let  $E$  be a singular fiber of  $\alpha$  and let  $\hat{E}$  be the normalization of  $E$ . Assume for the sake of proof by contradiction that  $E$  is totally geodesic. According to Lemma 6,

$$E \cdot E = \frac{1}{2}e(\hat{E}) + 2\delta^{\text{an}}(E)$$

and moreover,  $g = g(\hat{E}) + \delta^{\text{an}}(E)$  and  $E \cdot E = 0$  since  $E$  is a fiber of the fibration, hence  $1 - g + 3\delta^{\text{an}}(E) = 0$ . Since we have shown that  $g = 19$  in Theorem 3, this leads to  $\delta^{\text{an}}(E) = 6$ . However, totally geodesic curves have simple crossings and computations of Lemma 6 and Proposition 10 show that if  $P_1, \dots, P_k$  are the singular points of  $E$  with  $b_i$  local branches at

$P_i$  then  $\sum_{i=1}^k \mu_{P_i} = \sum_{i=1}^k (2\delta_{P_i}^{\text{an}} - \delta_{P_i}^{\text{top}}) = \sum_{i=1}^k (b_i - 1)^2 \leq 3$  by Corollary 2, formula (12). The only possibility is  $k \leq 3$  and  $b_i = 2$  for all  $i$  but then  $\delta^{\text{an}}(E) = \frac{1}{2} \sum_{i=1}^k b_i(b_i - 1) \leq 3$ , a contradiction.  $\square$

**5.6.** In his PhD thesis [Li], R. Livné constructed two-ball quotients by taking branched coverings of some generalized universal elliptic curves with level structure, and by construction, these surfaces admit a fibration onto a curve. The Albanese fibration of the Cartwright-Steger surface does not appear in the same fashion, but one can exhibit another (rational) fibration from  $X$  onto  $\mathbb{P}_{\mathbb{C}}^1$  appearing in a quite similar way to Livné's. Its generic fiber has genus 109, and  $\sum_{i=1}^4 C_i$ , with  $C_i$  given in §2.4, is one of the fibers (cf. [CKY, §6]).

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