MATH 402 Homework 5

Due Friday October 12, 2018

Exercise 1. This exercise is about **Poincaré lines and the Poincaré distance formula**.

a. *[2 pts]* Recall that hyperbolic distance is defined by the formula

$$d_P(P, Q) = \left| \ln \left(\frac{(PS)}{(PR)} \frac{(QR)}{(QS)} \right) \right|.$$

Draw a picture showing P, Q, R, and S.

b. *[5 pts]* Show that $d_P(P, Q) = 0$ if and only if $P = Q$. (Hint: to show "$\Rightarrow$" consider the ratios $\frac{PS}{QS}$ and $\frac{PR}{QR}$. Suppose $PS < QS$; what does this tell you about PR and QR?)

c. *[3 pts]* If $Q = O$ is the centre of the unit circle, simplify the formula for $d_P(P, Q)$.

Exercise 2. This exercise is about **Klein lines and the Klein model**.

a. *[2 pts]* Draw a picture of the Klein disk, in which you have a line ℓ and a point P not on ℓ. Draw the two *limiting parallels* to ℓ through P, call them m and m'.

b. *[4 pts]* Draw a perpendicular line to ℓ through P. (You may need to draw some other (Euclidean) lines to show that this line is perpendicular.) Label the intersection point by Q. Label the *angle of parallelism* to ℓ at P.

c. *[9 pts]* Start a new drawing, this time just drawing ℓ, P, and one limiting parallel m from part (a). Working from this drawing, prove that there is no Klein line which is perpendicular to both ℓ and m.

Exercise 3. Let S be any set, and let $f, g : S \to S$ be any two functions. Recall that we say that g is the *inverse* of f (and write $f^{-1} := g$) if for every $s \in S$ we have

$$f(g(s)) = s; \qquad g(f(s)) = s.$$

A function f which has an inverse is called *invertible*.

a. *[4 pts]* A function $f : S \to S$ is called *bijective* if it is both injective ('one-to-one') and surjective ('onto'). Prove that a bijective function f must have a unique inverse.

b. *[4 pts]* Check that if f and g are invertible with inverses f^{-1}, g^{-1}, and if $h = f \circ g$, then h is invertible with $h^{-1} = g^{-1} \circ f^{-1}$.

c. *[4 pts]* In particular, we defined a *transformation* to be a bijection of the plane, so it follows immediately from the above that a transformation has an inverse. Recall that a transformation is called an *isometry* if it preserves length. Prove that if f is an isometry, then its inverse is also an isometry.

d. *[4 pts]* Prove that if f and g are isometries, then $f \circ g$ is an isometry.

e. *[4 pts]* Combine the last two parts of the exercise, and use the fact that composition of functions is associative, to show that the set of isometries is a group. (You may need to review the definition of a group! Make sure you address each group axiom in your solution.)

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.