MATH 595 Thursday 15 February

$\check{\text{E}}$xt groups and sheaves

(1) **Chapter III, Exercise 6.1.**

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}'$ and $\mathscr{F}''$ be $\mathcal{O}_X$ modules. An *extension of $\mathscr{F}''$ by $\mathscr{F}'$* is a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''.$$

Two extensions are *isomorphic* if there is an isomorphism of the short exact sequences in which the maps $\mathscr{F}' \to \mathscr{F}'$ and $\mathscr{F}'' \to \mathscr{F}''$ are both identity morphisms. Let E be the set of isomorphism classes of extensions.

In this exercise, you will show that there is a bijection between E and $\mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}')$.

(a) We construct a map $\theta : E \to \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}')$ as follows: given $\xi = \left[0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0\right]$, the long exact sequence associated to $\mathrm{Hom}(\mathscr{F}'', \cdot)$ gives a map

$$\delta : \mathrm{Hom}(\mathscr{F}'', \mathscr{F}'') \to \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}').$$

We set $\theta(\xi) = \delta(\mathrm{id})$.
Prove that θ is well-defined on isomorphism classes of extensions.

(b) To an element $\alpha \in \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}')$, together with a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{I} \to \mathscr{G} \to 0$$

with $\mathscr{I}$ injective, construct an extension $\phi(\alpha)$, whose middle term is a certain fibre product of $\mathscr{I}$ and $\mathscr{F}''$ over $\mathscr{G}$.

(c) Show that $\theta(\phi(\alpha)) = \alpha$, and conclude that θ is surjective.

(d) To show that θ is injective, prove that every extension is isomorphic to one of the form $\phi(\alpha)$ for some α.

(2) **Chapter III, Exercise 6.10.**

Let $f : X \to Y$ be a finite morphism of noetherian schemes. For $\mathscr{G} \in \mathrm{QCoh}(Y)$, it is not hard to see that $\mathscr{H}om(f_*\mathcal{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathcal{O}_X$-module. Hence (because f is affine), it corresponds uniquely to a quasi-coherent $\mathcal{O}_X$-module $f^!\mathscr{G}$, with the defining property

$$f_*f^!(G) = \mathscr{H}om_Y(f_*(\mathcal{O}_X), \mathscr{G}).$$

(a) Write down a natural map $f_*f^!\mathscr{G} \to \mathscr{G}$.

(b) For any $\mathscr{M}, \mathscr{N} \in \mathrm{QCoh}(X)$, write down a natural map

$$f_*\mathscr{H}om_X(\mathscr{M}, \mathscr{N}) \to \mathscr{H}om_Y(f_*\mathscr{M}, f_*\mathscr{N}).$$

(c) Take $\mathscr{F} \in \mathrm{Coh}(X)$. Use the two maps above to give a natural map

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \to \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G}),$$

and prove (for example by checking over affine pieces of Y) that it is an isomorphism.

(d) Use the map from (b) and properties of δ-functors to prove that for every $i \geq 0$ there is a natural map

$$\phi_i : \mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G}).$$

(e) Assume that X and Y are separated, $\mathrm{Coh}(X)$ has enough locally free sheaves, and $f_*(\mathcal{O}_X)$ is locally free on Y (this means f is *flat*). Prove that in this case ϕ_i is an isomorphism for every i, for every $\mathscr{F} \in \mathrm{Coh}(X)$, and for every $\mathscr{G} \in \mathrm{QCoh}(Y)$. (Hint: do this by induction on i. The case $i = 0$ is easy using what you've done so far. Before doing the induction step, first check that the claim holds for all i as long as $\mathscr{F}$ is free, and then check that it holds as long as $\mathscr{F}$ is locally free. Finally, to do the induction step, write a short exact sequence showing $\mathscr{F}$ as a quotient of a locally free sheaf. You get another short exact sequence by pushing forward along f_*, because f_* is exact in this case. Now you get two long exact sequences, corresponding to the two sides of ϕ_i. Use induction and the five lemma.)