From last time: • The linearization of $f: \mathbb{R}^2 \to \mathbb{R}$ at (a,b) is:

$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

• f is differentiable at (a,b) if

$$\boxed{1}. \quad \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(x+\Delta x, y+\Delta y) - L(x+\Delta x, y+\Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.$$

Today: the chain rule.

· Recall: given $\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$, $h = g \circ f$, $h(t) = g(f(t))$, we can compute $h'(t)$ in terms of g and f:

$$h'(t) = g'(f(t))\, f'(t).$$

· Multivariable set-up

Fix $f: \mathbb{R}^2 \to \mathbb{R}$ (assume differentiable)

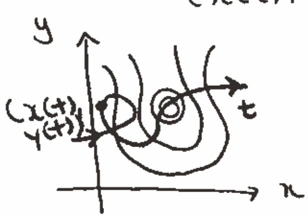
$x, y: \mathbb{R} \to \mathbb{R}$

and let $h(t) = f(x(t), y(t))$, so $h: \mathbb{R} \to \mathbb{R}$.

Can we calculate $h'(t)$ in terms of f, x, y?

e.g. $f(x,y) =$ ~~pos~~ temperature in terms of position.

$(x(t), y(t))$ - curve parametrizing path of an ant.



$h(t) =$ temperature of ant at time t.

↳ Chain rule:

$$\frac{dh}{dt}(x(t), y(t)) = \frac{\partial \cancel{h} f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt}(t) + \frac{\partial \cancel{h} f}{\partial y}(x(t), y(t)) \frac{dy}{dt}(t).$$

Sketch of proof:

$$\frac{\cancel{h}f(x(t+\varepsilon), y(t+\varepsilon)) - \cancel{h}f(x(t), y(t))}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \frac{dh}{dt}(t).$$

But also:

$$\frac{f(x(t+\varepsilon), y(t+\varepsilon)) - f(x(t),y(t))}{\varepsilon} = \frac{f(x(t+\varepsilon), y(t+\varepsilon)) - f(x(t), y(t+\varepsilon))}{\varepsilon}$$

$$+ \frac{f(x(t), y(t+\varepsilon)) - f(x(t),y(t))}{\varepsilon}. \qquad (*)$$

Now $$\frac{f(x(t),y(t+\varepsilon)) - f(x(t),y(t))}{\varepsilon} = \frac{g(y(t+\varepsilon)) - g(y(t))}{\varepsilon}$$

$$= \frac{g(y(t) + \overbrace{(y(t+\varepsilon)-y(t))}^{\varepsilon'}) - g(y(t))}{\underbrace{y(t+\varepsilon)-y(t)}_{\varepsilon'}} \cdot \underbrace{\frac{y(t+\varepsilon)-y(t)}{\varepsilon}}$$

$$\downarrow \ \varepsilon \to 0 \ (\varepsilon' \to 0) \qquad\qquad \downarrow \ \varepsilon \to 0$$

$$g'(y(t)) \qquad\qquad y'(t).$$

$$= \frac{\partial f}{\partial y}(x(t), y(t)).$$

So $$(*) \xrightarrow[\varepsilon \to 0]{} \frac{\partial f}{\partial x}(x(t),y(t))\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t),y(t))\frac{dy}{dt}(t). \quad \square.$$

<u>Short-hand notation:</u> evaluate at t_0.

- $$\frac{dh}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

evaluate at $(x(t_0), y(t_0))$

- $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

here $z = h(t)$
$= f(x,y)$ } not literally the same function (confusing!)

<u>Example:</u> $x(t) = \sqrt{2}\cos t$, $y(t) = \sqrt{2}\sin t$

$f(x,y) = (x+y^2)^2$, $h(t) = f(x(t), y(t))$.

Find $\frac{dh}{dt}(\pi/4)$.

Chain Rule:

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

at $(\pi/4)$; at $(x(\pi/4), y(\pi/4)) = (1, 1)$; at $\pi/4$.

- $f_x(x,y) = 2(x+y^2) \Rightarrow f_x(1,1) = 4.$
- $f_y(x,y) = 4y(x+y^2) \Rightarrow f_y(1,1) = 8$

$\frac{dx}{dt} = -\sqrt{2}\cos(t) \Rightarrow x'(\pi/4) = -1$

$\frac{dy}{dt} = \sqrt{2}\sin(t) \Rightarrow y'(\pi/4) = 1.$

$\therefore \frac{dh}{dt}(\pi/4) = 4(-1) + 8(1) = 4.$

Rmk: Also, can write $h(t) = (\sqrt{2}\cos t + 2\sin^2 t)^2$, so we can calculate $\frac{dh}{dt}$ directly to double check.

$\boxed{2}$ **Question:** let $W(t) = F(u(t), v(t))$.

Assume $u(1) = 0$, $v(1) = 3$, $u'(1) = 2$, $v'(1) = 0$

$F_u(0,3) = 1$; $F_u(2,0) = -1$, $F_v(2,0) = 8$.

Find $\frac{dW}{dt}(1)$.

§ GENERALIZATIONS.

1) curves in $\mathbb{R}^n$: $f: \mathbb{R}^n \to \mathbb{R}$

$x_i: \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, \ldots, n$.

$\leadsto h(t) = f(x_1(t), x_2(t), \ldots, x_n(t))$.

$$\frac{dh}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}.$$

2) x, y depend on two variables instead of 1:

$x: \mathbb{R}^2 \to \mathbb{R}$ $\qquad$ $y: \mathbb{R}^2 \to \mathbb{R}$.

$(s,t) \mapsto x(s,t)$.

↳ $h(s,t) = f(x(s,t), y(s,t))$.

↳ $\dfrac{\partial h}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$; $\dfrac{\partial h}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$.

<u>Example:</u> Suppose $x(s,t)$, $y(s,t)$ are differentiable.

and consider $f : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto \cos(x+y)$. Let $h(s,t) = f(x(s,t), y(s,t))$

$\boxed{2}$ What can you say about $\cancel{\frac{\partial f}{\partial}}$ $\dfrac{\partial h}{\partial s}(\pi,\pi)$?