[1] Is there a point on the graph $z = \sqrt{x^2+y^2}$ that's closest to the point $P = (4,2,0)$? Furthest?

<u>Recall</u> - <u>the extreme value theorem:</u>

Let $f : D \to \mathbb{R}$ be continuous; D closed & bounded.
Then f attains a maximum value at some point $P \in D$, and

either · $P \in \partial D$ = boundary of D

· P is a critical point for f.

<u>Today</u>: How can we find the maximum value for f over ∂D?

<u>Example</u>: Does $f(x,y) = x^2 - y^2$ have a maximum value on $D = x^2+y^2 \le 4$? What is it?

· has max value by EVT, since D is closed & bounded.

· <u>critical points</u>: $\nabla f = \langle 2x, -2y \rangle = \langle 0, 0 \rangle$ at $(0,0)$.

· $f(0,0) = \underline{0}$.

· <u>boundary points</u>: $\partial D = \{ x^2+y^2 = 4 \}$ $\quad x \in [-2,2]$

<u>Method 1</u>

$y^2 = 4 - x^2$

$\Rightarrow$ on ∂D, $f(x,y) = x^2 - y^2 = 2x^2 - 4$. $\quad x \in [-2,2]$

So we need to find max of $g(x) = 2x^2 - 4$ on $[-2,2]$

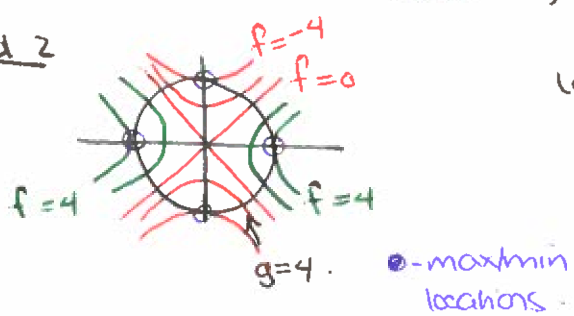
· critical point(s): $g'(x) = 4x = 0 \iff x = 0$.

$g(0) = f(0, \pm 2) = \underline{-4}$

· end points $g(\pm 2) = f(\pm 2, 0) = (\pm 2)^2 - (0)^2 = \underline{4}$

So the maximum value of f is 4, at $(\pm 2, 0)$.

<u>Method 2</u>



let $g(x,y) = x^2 + y^2$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$\nabla f(x,y) = \langle 2x, -2y \rangle$

max values at $(\pm 2, 0)$

$\nabla g = \langle \pm 4, 0 \rangle$

$\nabla f = \langle \pm 4, 0 \rangle$

· Min value at $(0, \pm 2)$ $\rightarrow \nabla g = \langle 0, \pm 4 \rangle$
$\nabla f = \langle 0, \mp 4 \rangle$

Exactly the locations where ∇g, ∇f point in the same direction ($\pm$).

Theorem (Lagrange multipliers) · discovered by Euler.

Assume $f, g : \mathbb{R}^2 \to \mathbb{R}$ have continuous first order partial derivatives.

If $f(x_0, y_0)$ is the maximum value of f on the level curve $\{g(x,y) = k\}$ then either

g is called the constraint / side condition.

- $\nabla g(x_0, y_0) = \langle 0, 0 \rangle$

OR
- $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some $\lambda \in \mathbb{R}$.

Note: the theorem doesn't guarantee a maximum exists, it just tells us where it's possible a maximum does exist.

- But if g is continuous, $\{g = k\}$ is closed, so if we can show it's bounded, a maximum exists by EVT.
- If not, sometimes we can use a geometric/physical argument.

Strategy: (1) Check that $\{g = k\}$ is bounded
OR use a geometric argument.

(2) Check that $\nabla g(x,y) \neq 0$ on $g = k$

(3) Find all x, y, λ s.t.

- $\nabla f(x,y) = \lambda \nabla g(x,y)$ ← two equations
- $g(x,y) = k$ ← one more equation.

(4) Calculate $f(x,y)$ $\forall$ (x,y) in (3).

(5) Pick the largest.

Example Find the max of $f(x,y) = x^2 - y^2$ on $x^2 + y^2 = 4$

(1) $\{g(x,y) = x^2 + y^2 = 4\}$ ← boundary of D from before.

• closed & bounded.

(2) • $\nabla g(x,y) = \langle 2x, 2y \rangle \neq \langle 0,0 \rangle$ on $g=4$.

(3) 3 equations:

$$\nabla f = \lambda \nabla g : \begin{cases} \cdot\ 2x = \lambda 2x & \Rightarrow \lambda = 1 \text{ or } x = 0 \\ \cdot\ -2y = \lambda 2y & \Rightarrow \lambda = -1 \text{ or } y = 0. \\ \cdot\ x^2 + y^2 = 4 & \Rightarrow \text{can't have both } x=0 \text{ and } y=0. \end{cases}$$

$\boxed{2}$

Solutions are • $x=0,\ y=\pm 2,\ \lambda = -1$

• $y=0,\ x=\pm 2,\ \lambda = 1$

(4) Evaluate: $f(0,\pm 2) = -4$

$f(\pm 2, 0) = 4$.

(5) Compare: max is 4, min is -4.

Theorem (Lagrange multipliers in 3 variables)

Assume f, g are functions of three variables with continuous first partial derivatives

Then if $f(x_0, y_0, z_0)$ is the max value of f on the level set $\{g(x,y,z) = k\}$ either

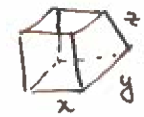
• $\nabla g(x_0, y_0, z_0) = \langle 0,0,0 \rangle$ OR • $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ for some $\lambda \in \mathbb{R}$.

Note: Similar theorems hold for minima.

$\boxed{2}$ **Example:** Assume that $f(x,y,z)$, $g(x,y,z)$ have continuous first partial derivatives. Suppose that f achieves its maximum value on the level set $\{g(x,y,z) = 3\}$ at P.

Which is <u>not</u> possible?

Example 4. Find the maximum volume of a box with surface area $6m^2$. 15.4



- Function $V(x,y,z) = xyz$
- constraint: $A(x,y,z) = 2xy + 2xz + 2yz = 6$.

  * but actually we have more constraints

  $x, y, z > 0$.

So we're looking for the maximum of V over

$$D = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} \cdot\, A(x,y,z) = 6 \\ \cdot\, x>0,\ y>0,\ z>0 \end{array} \right\}$$

[1] Can we find a max ?

System of equations :

- $\begin{cases} \cdot\ yz = \lambda(2y+2z) \\ \cdot\ xz = \lambda(2x+2z) \\ \cdot\ xy = \lambda(2x+2y) \end{cases}$ AND $x, y, z > 0$.
- $2xy + 2xz + 2yz = 6$

$\cancel{\Rightarrow 4\lambda(2x+y+z) = 6,\ x = y = z}$.

$\cancel{\Rightarrow 12\lambda = 6,\ x = \frac{1}{2\lambda} = y = z}$.

$\cancel{yz = \lambda(2y+2z) \Rightarrow \frac{1}{4\lambda^2} = \lambda(\frac{1}{\lambda}+\frac{1}{\lambda}) = 2}$

$\cancel{\Rightarrow 1 = 8\lambda^3}$.

$$\Rightarrow \frac{1}{2\lambda} = \frac{1}{x} + \frac{1}{y} = \frac{1}{x} + \frac{1}{z} = \frac{1}{y} + \frac{1}{z}$$

$$\Rightarrow \frac{1}{x} = \frac{1}{y} = \frac{1}{z}, \text{ so } x = y = z.$$

So $2x^2 + 2x^2 + 2x^2 = 6x^2 = 6$

$\Rightarrow x = \pm 1$

But $x > 0 \Rightarrow x = y = z = 1$

volume is 1.

(Note: we never needed to solve for λ)