Last time: iterated integrals

Let $D = [0, 2] \times [-3, 1]$. Find $\iint (3x^2 + 3y^2) dA$.

(a) -12

(b) 42

(c) 88

(d) Some other number

(e) I don't know

(If you're done, try integrating using the opposite order of integration to what you used the first time. You should get the same answer.)

Solution

$$\begin{aligned}
\iint_D (3x^2+3y^2)\,dA &= \int_0^2 \int_{-3}^{1} (3x^2+3y^2)\,dy\,dx \\
&= \int_0^2 \left[3x^2y + y^3\right]_{-3}^{1} dx \\
&= \int_0^2 \left[(3x^2+1) - (-9x^2-27)\right] dx \\
&= \int_0^2 12x^2 + 28 \; dx \\
&= [4x^3 + 28x]_0^2 \\
&= 32 + 56 = 88.
\end{aligned}$$

Solution (opposite order)

$$\begin{aligned}
\iint_D (3x^2 + 3y^2) dA &= \int_{-3}^{1} \int_{0}^{2} (3x^2 + 3y^2) dx dy \\
&= \int_{-3}^{1} \left[x^3 + 3xy^2\right]_0^2 dy \\
&= \int_{-3}^{1} \left[8 + 6y^2\right] dy \\
&= \left[8y + 2y^3\right]_{-3}^{1} \\
&= (8 + 2) - (-24 - 54) \\
&= 88.
\end{aligned}$$

Recall: Fubini's Theorem

Theorem

Let f be a continuous function on $D = [a, b] \times [c, d]$. Then

$$\iint_D f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx.$$

More generally, this is true if f is bounded and is continous except at a finite number of smooth curves, provided that the iterated integrals exist.

Regions of Type I

We say $D \subset \mathbb{R}^2$ is of Type I if it is of the form

$$D = \{(x, y) \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(y)\},$$

where $g_1, g_2 : [a, b] \to \mathbb{R}$ are continous functions.

Theorem

Let $f(x, y)$ be a continous function on a region D of type I as above. Then

$$\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx.$$

Practice with regions of type II

Recall the region D enclosed by the lines $x = 0$, $y = 1$, and the curve $y = x^2$.

To show that D is a region of type II, we need to find numbers c and d and continuous functions h_1, h_2 on the interval $[c, d]$ such that

$$D = \{(x, y) \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(y)\}.$$

(a) I don't know what to do.

(b) I'm working on it.

(c) I have answers, but they don't match with my neighbour's.

(d) We agree.

Solution

To find the interval c and d we look at the "shadow" produced by the region D on the y-axis. (Pretend a big light is shining from the far right side.)

So we see that $[c, d] = [0, 1]$.

Now given a point y_0 in this interval, what values can x take?

x must be larger than $\sqrt{y_0}$ and smaller than 1.

So $h_1(y) = \sqrt{y}$ and $h_2(y) = 1$.

Integrating over a region of type II

Let $D = \{(x, y) \mid 0 \leq y \leq 1 \text{ and } \sqrt{y} \leq x \leq 1\}$. How would you find the area of D? Fill in the blanks in the following formula:

$$\text{Area of } D = \int_{?}^{?} \int_{?}^{?} ?\ d?\ d?.$$

(a) I don't know what to do.

(b) I'm working on it.

(c) I have answers, but they don't match with my neighbour's.

(d) We agree.

Solution

$D = \{(x, y) \mid 0 \leq y \leq 1 \text{ and } \sqrt{y} \leq x \leq 1\}$

The area of D is $\iint_D 1 dA$; since D is of type II we have

$$\iint_D 1 dA = \int_0^1 \int_{\sqrt{y}}^1 1 dx dy.$$

Practice with integrating over polar rectangles

Let $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4 \text{ and } 0 \leq y\}$ as before. What is

$$\iint_D y dA?$$

(a) 0

(b) $\frac{14}{3}$

(c) 3

(d) $3\pi y^2$

(e) I don't know.

Solution

Since D is a polar rectangle of the form with $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$, we have

$$\begin{aligned}
\iint_D y dA &= \int_0^{\pi} \int_1^2 (r \sin\theta) r dr d\theta \\
&= \int_0^{\pi} \int_1^2 r^2 \sin\theta dr d\theta \\
&= \int_0^{\pi} [\frac{1}{3} r^3 \sin\theta]_1^2 d\theta \\
&= \int_0^{\pi} \frac{7}{3} \sin\theta d\theta \\
&= [\frac{-7}{3} \cos\theta]_0^{\pi} \\
&= \frac{14}{3}.
\end{aligned}$$