Last time: integrating in polar coordinates

The equation $r = \cos 2\theta$ traces out a "four-leafed rose" as θ varies. In particular, a leaf or petal of the rose is the region enclosed one loop of the curve, given by

$$D = \{(r, \theta) \mid \frac{-\pi}{4} \leq \theta \frac{\pi}{4}, 0 \leq r \leq \cos 2\theta\}.$$

Sketch the curve. Which formula can be used to calculate the area of this leaf?

(a) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} dr d\theta$

(b) $2 \int_0^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r dr d\theta$

(c) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r dr d\theta$

(d) $\int_0^{\cos 2\theta} \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} r d\theta dr$

(e) I don't know.

Correct answer: (b) (because of symmetry of the leaf) or (c).

Solution

$$
\begin{aligned}
\text{Area(D)} &= \iint_D dA \\
&= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r \;\, dr d\theta \\
&= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left[\frac{1}{2} r^2\right]_0^{\cos 2\theta} d\theta \\
&= \frac{1}{2} \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta d\theta \\
&= \frac{1}{4} \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 4\theta) d\theta \\
&= \frac{1}{4} \left[\theta + \frac{1}{4} \sin 4\theta\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}
\end{aligned}
$$

Practice with triple integrals

We wrote

$$E = \{(x, y, z) \mid (x, z) \in D_2, x^2 + z^2 \leq y \leq 4\},$$

where $D_2 = \{(x, z) \mid x^2 + z^2 \leq 4\}$. so

$$\begin{aligned} V(E) &= \iiint_E dV \\ &= \iint_{D_2} \int_{x^2+z^2}^{4} dy \; dA \\ &= \iint_{D_2} [y]_{x^2+z^2}^{4} \, dA \\ &= \iint_{D_2} 4 - x^2 - z^2 dA. \end{aligned}$$

Practice with triple integrals

So we need to calculate $\iint_{D_2} 4 - x^2 - z^2 dA$. We could write D_2 as a region of type I or II, but it is easier to use polar coordinates (in the xz-plane):

$$x = r\cos\theta, \qquad z = r\sin\theta;$$

$$\iint_{D_2} 4 - x^2 - z^2 dA = \int_0^{2\pi}\int_0^2 (4 - r^2)r\ drd\theta$$

$$\int_0^{2\pi}\left[2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta$$

$$\int_0^{2\pi} 4d\theta = 8\pi.$$

Practice with triple integrals

E is the region bounded by the planes $x = 2, y = 2, z = 0$ and $x + y - 2z = 2$, and D is its triangular shadow in the xy-plane.

Fill in the blanks, and discuss with your neighbour:

$$D = \{(x, y) \mid ? \leq x \leq ?, ?? \leq y \leq ??\}.$$

(a) We're working on it.
(b) We're stuck.
(c) We have answers, but they're different and we don't know who is right.
(d) We have the same answer.

Solution

$$D = \{(x, y) \mid 0 \leq x \leq 2,\ 2 - x \leq y \leq 2\}.$$

Practice with triple integrals

Now find $u_1(x, y)$ and $u_2(x, y)$ such that

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}.$$

(a) We're working on it.

(b) We're stuck.

(c) We have answers, but they're different and we don't know who is right.

(d) We have the same answer.

Solution

$$u_1(x, y) = 0$$
$$u_2(x, y) = \frac{x}{2} + \frac{y}{2} - 1.$$

So if f is a function on E, then

$$\iiint_E f dV = \int_0^2 \int_{2-x}^2 \int_0^{\frac{x}{2}+\frac{y}{2}-1} f \, dzdydx.$$