Last time: cylindrical and spherical coordinates

Recall that (x, y, z) and (ρ, θ, ϕ) are related by

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi.$$

Consider the solid C lying between the half-cone $z = \sqrt{x^2 + y^2}$ and the half-sphere $z = \sqrt{9 - x^2 - y^2}$. Sketch C and write it in spherical coordinates:

$$C = \{(\rho, \theta, \phi) \mid \square \leq \theta \leq \square,\ \square \leq \phi \leq \square,\ \square \leq \rho \leq \square\}.$$

What is the sum of the numbers in the boxes?

(a) $9 + \frac{5\pi}{4}$

(b) $3 + \frac{5\pi}{4}$

(c) $3 + 3\pi$

(d) $9 + 3\pi$

Solution

$C = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{4},\ 0 \leq \rho \leq 3\}.$
(So none of the options given above were correct, sorry for the typo...)

Announcements

The American Society of Mechanical Engineers at the University of Illinois wants you to sign up for their talent show. Here is a link to their poster:
Poster.

Recall: Integrating in spherical coordinates

Let B be a "spherical wedge":

$$B = \{(\rho, \theta, \phi) \mid \alpha \le \theta \le \beta,\ a \le \rho \le b,\ c \le \phi \le d\}.$$

Let $f : B \to \mathbb{R}$ be a continuous function. Then

$$\iiint_B f dV =$$

$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \ d\rho d\theta d\phi.$$

Example

Let's find the volume of the solid C from the first question.

$$C = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{4},\ 0 \leq \rho \leq 3\}$$

So we have

$$\begin{aligned}
V(C) &= \iiint_C dV \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^3 \rho^2 \sin\phi \ d\rho d\theta d\phi \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \left[\frac{1}{3}\rho^3 \sin\phi\right]_0^3 \ d\theta d\phi \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} 9 \sin\phi \ d\theta d\phi \\
&= \int_0^{\frac{\pi}{4}} 18 \sin\phi \ d\phi = 9\pi(2 - \sqrt{2}).
\end{aligned}$$

Practice with Jacobians

Find the Jacobian for each of the examples (1), (2), (3). Which is the largest?

(a) (1)
(b) (2)
(c) (3)
(d) It's a tie.
(e) I don't know how to do this.

Solution

(1)

$$T(u,v) = 4u + \frac{1}{2}v$$
$$\frac{\partial(x,y)}{\partial(u,v)} = 4 \times \frac{1}{2} = 2.$$

(2)

$$T(u,v) = \frac{1}{\sqrt{2}}(u+v, -u+v)$$
$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}} \times \left(\frac{1}{\sqrt{2}}\right) = 1.$$

(Note there is a sign change from the example done in class.)

(3)

$$T(u,v) = (u+v, v)$$
$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \times 1 - 1 \times 0 = 1.$$

Change of variables — why does it work?

We calculate $\iint_{T(D)} f(x,y)dA$ by dividing D into small boxes $\square$ of area ΔA.

Then $T(D)$ is divided into small parallelograms $T(\square)$ of area $|\frac{\partial(x,y)}{\partial(u,v)}|\Delta A$.

We should choose test points (x^*, y^*) in each $T(\square)$. We do this by choosing points (u^*, v^*) in $\square$, and taking $(x^*, y^*) = T(u^*, v^*)$.

Then to compute the integral, we take the sum over all the parallelograms of the contribution

$$f(x^*, y^*) \cdot \text{Area}(T(\square)) = f(T(u^*, v^*)) \cdot |\frac{\partial(x,y)}{\partial(u,v)}|\Delta A,$$

and taking the limit as the number of boxes goes to infinity.

But this is just the integral $\iint_D f(T(u,v))|\frac{\partial(x,y)}{\partial(u,v)}|dA$.