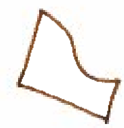
Monday, April 8, 2019

Review: integrating vector fields over curves.

[2] [See slides for example & solution]

Today: Green's theorem.

Recall: · A path is a piecewise smooth curve.



Fundamental theorem of calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Fundamental theorem of line integrals.

C is a path from A to B:

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A).$$

$$\parallel$$

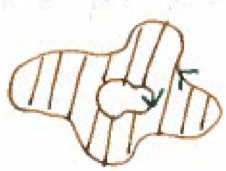
$$\int_C f_x\,dx + f_y\,dy$$

Derivative on the left — Boundary on the right

Today: integrate a "derivative" over a 2d region B $\leftrightarrow$ integrate the original term over the boundary curve ∂B.

Assumptions. [on slide]

- $\vec{F} = \langle P, Q \rangle$ has continuous first order partial derivatives on an open set $D \subset \mathbb{R}^2$.
- $B \subset D$ is "nice":
  - we can integrate over B
  - ∂B is one or more simple closed paths.
- orient ∂B so that B is always on the left.



Theorem: [Green's Theorem]

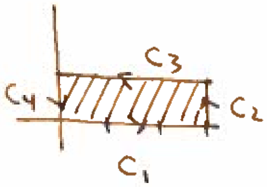
$$\iint_B \underbrace{(Q_x - P_y)}_{\text{derivative}}\,dA = \int_{\partial B} P\,dx + Q\,dy$$

boundary $\rightarrow$ $= \int_{\partial B} \vec{F} \cdot d\vec{r}$

(later, we'll see two more theorems with the same structure)

Warning: Make sure $\vec{F}$ is defined on all of B.

Example: Find $\int_C \underbrace{xy}_{P}\,dx + \underbrace{x^2}_{Q}\,dy$, where C is the rectangle with vertices $(0,0)$, $(3,0)$, $(3,1)$, $(0,1)$.



$B = [0,3] \times [0,1]$. Nice!

By Green's theorem:

$$\int_C xy\,dx + x^2\,dy = \iint_B \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\,dA$$

$$= \int_0^3 \int_0^1 2x - x\,dy\,dx = \int_0^3 \int_0^1 x\,dy\,dx$$

$$= \int_0^3 x\,dy = \left[\tfrac{1}{2}x^2\right]_0^3 = 9/2.$$

Theorem: Area of $B = \underset{Ⓐ}{\int_{\partial B} x\,dy} = \underset{Ⓑ}{-\int_{\partial B} y\,dx} = \underset{Ⓒ}{\tfrac{1}{2}\left(\int_{\partial B} x\,dy - y\,dx\right)}$

proof of Ⓒ: By Green's theorem,

$$\frac{1}{2}\left(\int_{\partial B} x\,dy - y\,dx\right) = \frac{1}{2}\left(\iint_B \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\,dA\right)$$

$$= \frac{1}{2}\iint_B 2\,dA = \iint_B dA = \text{Area of } B. \quad \square$$

(A) and (B) are similar.

1. Use (C) to find the area of the disk $B_r = \{x^2+y^2 \le r^2\}$.

2. Let $\vec{F} = \langle P,Q \rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

Recall that $P_y = Q_x$. Which argument is correct?

(A) on C_r, $\langle P,Q \rangle = \langle -y/r^2, x/r^2 \rangle$

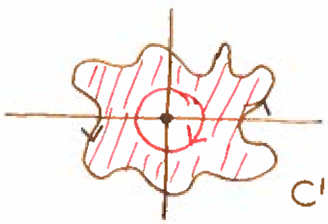
$$\Rightarrow \int_{C_r} \vec{F}\cdot d\vec{r} = \frac{1}{r^2}\int_{C_r} x\,dy - y\,dx = \frac{2\pi r^2}{r^2} = 2\pi.$$

(B) By Green's theorem

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \iint_{B_r} (Q_x - P_y)\,dA = \iint_{B_r} 0\, dA = 0.$$

Another example: With $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ as before,

let C' be any simple closed curve in $\mathbb{R}^2$ enclosing $(0,0)$.



What is $\int_{C'} \vec{F}\cdot d\vec{r}$?

We can't directly use Green's theorem, because $\vec{F}$ isn't defined at $(0,0)$.

Instead, choose $r>0$ small enough and consider $-C_r$, so that $C' \cup (-C_r)$ forms the boundary of a region B.

[L] Now use Green's theorem to calculate $\int_{C'} \vec{F}\cdot d\vec{r}$. [see slides].

Recall Theorem: If D is simply connected and $\vec{F}$ satisfies $P_y = Q_x$, then $\vec{F}$ is conservative.

We can prove this, using Green's theorem.

Recall $\vec{F}$ is conservative $\iff$ the $\int_C \vec{F}\cdot d\vec{r}$ is path independent

$\iff \int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve C.

Note that any closed curve C can be broken into simple closed curves

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \sum \pm \int_{C_i} \vec{F}\cdot d\vec{r}$$

$\nwarrow$ simple

So it's enough to show $\int_C \vec{F}\cdot d\vec{r} = 0$ for C a simple closed curve in D.

Since D is simply connected, $C = \partial B$ for $B \subset D$.

So by Green's Theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_B (Q_x - P_y)\, dA = \iint_B 0\, dA = 0 \qquad \square$$

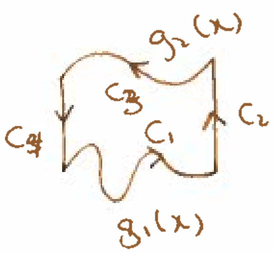
Why is Green's theorem true?

$$\int_C P\,dx + Q\,dy = \iint_B (Q_x - P_y)\, dA.$$

↳ it's enough to prove that $\displaystyle\int_C P\,dx = -\iint_B \frac{\partial P}{\partial y}\, dA \quad (*)$

and $\displaystyle\int_C Q\,dy = \iint_B \frac{\partial Q}{\partial x}\, dA. \quad (**)$

Let's show that $(*)$ is true for a region of type I.



$$B = \{ (x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x) \}.$$

$$\Rightarrow -\iint_B \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int_a^b P(x, g_2(x)) - P(x, g_1(x))\, dx$$

(by F.T.C.)

On the other hand

$$\int_C P\,dx = \int_{C_1} P\,dx + \underbrace{\int_{C_2} P\,dx}_{=0} + \int_{C_3} P\,dx + \underbrace{\int_{C_4} P\,dx}_{=0}$$

(because C_2, C_4 are vertical, and $\langle P, 0\rangle$ is only in the x-direction)

$\int_{C_1} P\,dx = \int_a^b P(t, g_1(t))\, dt$ (using parametrization of C_1 given by $\vec{r}(t) = \langle t, g_1(t)\rangle$)

$\int_{C_3} P\,dx = -\int_a^b P(t, g_2(t))\, dt.$

So the two sides are equal.

Likewise we prove that $\displaystyle\int_C Q\,dy = \iint_B \frac{\partial Q}{\partial x}\, dA$ for B of type II.

So we need to divide our region B into small regions that are both of type I AND type II