Last time: Green's theorem

Let $D \subset \mathbb{R}^2$ be such that ∂D is formed of one or more simple closed curves. Suppose $\mathbf{F} = \langle P, Q \rangle$ is a vector field such that P and Q have continuous first order partial derivatives. Then

$$\iint_D (Q_x - P_y) dA = \int_{\partial D} P dx + Q dy = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}.$$

Consider the picture on the chalkboard with paths C_1, C_2, C_3. Suppose that $Q_x = P_y$ on D and also suppose that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 3$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$. What is $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$?

(a) 0

(b) 2

(c) 4

(d) We don't have enough information.

Solution

Fill in the space surrounded by the curves C_1, C_2, C_3 to get a region D with boundary $\partial D = (-C_1) \cup (-C_2) \cup C_3$.

So we see that

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}.$$

On the other hand, by Green's theorem,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y dA,$$

but since $P_y = Q_x$ this is zero. Combining, we see that

$$0 = -\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = -3 - 1 + \int_{C_3} \mathbf{F} \cdot d\mathbf{r},$$

so $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4$.

Practice with curl

Find curl$\langle e^x, z\cos y, \sin y\rangle$. How many of the three components are 0?

(a) 0

(b) 1

(c) 2

(d) 3

(e) I don't know.

curl$\langle e^x, z\cos y, \sin y\rangle = \langle \cos y - \cos y, 0 - 0, 0 - 0\rangle = \langle 0, 0, 0\rangle$

Practice with curl

Find curl$\langle P(x,y), Q(x,y), 0\rangle$.

Solution

$$\begin{aligned}\text{curl}\langle P(x,y), Q(x,y), 0\rangle &= \langle \frac{\partial 0}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial 0}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\rangle \\ &= \langle 0, 0, Q_x - P_y\rangle.\end{aligned}$$

Practice with conservative vector fields and curl

Is $\mathbf{F}(x, y, z) = \langle xz, xyz, -y^2 \rangle$ conservative?
Is $\mathbf{G}(x, y, z) = \langle e^x, z \cos y, \sin y \rangle$ conservative?

(a) Yes and yes.

(b) Yes and no.

(c) No and yes.

(d) No and no.

(e) Not enough information.

curl$\mathbf{F} \neq \mathbf{0}$, so $\mathbf{F}$ is not conservative.
curl$\mathbf{G} = \mathbf{0}$ on *all of* $\mathbb{R}^3$, so $\mathbf{G}$ is conservative.