Tuesday, January 22 $\ast\ast$ *Projections, distances, and planes.*

1. Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 1\mathbf{j}$.

   (a) Calculate $\text{proj}_{\mathbf{b}}\mathbf{a}$ and draw a picture of it together with $\mathbf{a}$ and $\mathbf{b}$.

   (b) The orthogonal complement of the vector $\mathbf{a}$ with respect to $\mathbf{b}$ is defined by

   $$\text{orth}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}.$$

   Calculate $\text{orth}_{\mathbf{b}}\mathbf{a}$ and draw two copies of it in your picture from part (a), one based at $\mathbf{0}$ and the other at $\text{proj}_{\mathbf{b}}\mathbf{a}$.

   (c) Check that $\text{orth}_{\mathbf{b}}\mathbf{a}$ calculated in (b) is orthogonal to $\text{proj}_{\mathbf{b}}\mathbf{a}$ calculated in (a).

   (d) Find the distance of the point $(1,1)$ from the line $(x, y) = t(2,-1)$. Hint: relate this to your picture.

2. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$. Use the definitions of $\text{proj}_{\mathbf{b}}\mathbf{a}$ and $\text{orth}_{\mathbf{b}}\mathbf{a}$ to show that $\text{orth}_{\mathbf{b}}\mathbf{a}$ is always orthogonal to $\text{proj}_{\mathbf{b}}\mathbf{a}$.

3. Find the distance between the point $P(3,4,-1)$ and the line $\mathbf{l}(t) = (2,3,-2) + t(1,-1,1)$. Hint: Consider a vector starting at some point on the line and ending at P, and connect this to what you learned in Problem 1.

4. Consider the equation of the plane $x + 2y + 3z = 12$.

   (a) Find a normal vector $\mathbf{n}$ to the plane. (Just look at the equation!)

   (b) Find where the x, y, and z-axes intersect the plane. Using this information, sketch the portion of the plane in the first octant where $x \geq 0, y \geq 0, z \geq 0$.

   (c) Using the points in part (b), find two non-parallel vectors that are parallel to the plane.

   (d) Using the dot product to check that the vectors you found in (c) are really orthogonal to $\mathbf{n}$.

   (e) Pick another normal vector $\mathbf{n}'$ to the plane and one of the points from (b). Use these to find an alternative equation for the plane. Compare this new equation to $x + 2y + 3z = 12$. How are these two equations related? Is it clear that they describe the same set of points (x, y, z) in $\mathbb{R}^3$?

5. *The Triangle Inequality.* Let $\mathbf{a}$ and $\mathbf{b}$ be any vectors in $\mathbb{R}^n$. The triangle inequality states that $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.

   (a) Give a geometric interpretation of the triangle inequality. (E.g. draw a picture in $\mathbb{R}^2$ or $\mathbb{R}^3$ that represents this inequality.)

   (b) Use what we know about the dot product to explain why $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$. This is called the Cauchy-Schwarz inequality.

   (c) Use part (b) to justify the triangle inequality. Hint: Start with the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and then use properties of the dot product and the Cauchy-Schwarz inequality to manipulate the right-hand side into looking like $|\mathbf{a}|^2 + |\mathbf{b}|^2$.