Last lecture (revision)

Let $A = (a_{ij})$ be an $n \times m$-matrix and let $a_j = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix}$ be the i^{th} column of A.

Suppose that the matrix equation $Ax = b$ has a solution.

- The equation $Ax = b$ has infinitely solutions if and only if $Ax = 0$ has infinitely solutions.
- Equivalently The equation $Ax = b$ has a unique solution if and only if $Ax = 0$ has a unique solution.
- Suppose that y is a solution of $Ax = b$. Then every solution of $Ax = b$ is of the form $x = y + tz$, where $Az = 0$ (i.e. $z \in \text{Null}(A)$) and $t \in \mathbb{R}$.
- The Null space of the matrix A is the set

$$\text{Null}(A) = \{ z \in \mathbb{R}^m : Az = 0 \}.$$

We will see today that $\text{Null}(A)$ is a vector space.

Definition A real vector space is a non–empty set V which has two operations vector addition $+$ and scalar multiplication $\cdot$ which have the following properties:

A1 (closure) $x + y \in V$.

A2 (associativity) $(x + y) + z = x + (y + z)$.

A3 (commutativity) $x + y = y + x$.

A4 (zero) There is an element $0 \in V$ such that $x + 0 = x$, for all $x \in V$.

A5 (negatives) For each $x \in V$ there is an element $x' \in V$ such that $x + x' = 0$.

S1 (closure) $r \cdot x \in V$.

S2 (distributivity) $r \cdot (x + y) = r \cdot x + r \cdot y$.

S3 (distributivity) $(r + s) \cdot x = r \cdot x + s \cdot x$.

S4 (commutativity) $(rs) \cdot x = r \cdot (s \cdot x)$.

S5 (identity) $1 \cdot x = x$.

for all $x, y, z \in V$ and all $r, s \in \mathbb{R}$.

We usually write rx rather than $r \cdot x$.

Examples of vector spaces–I

Let A be an $n \times m$ matrix and let $V = \text{Null}(A)$.

Claim : V is a vector space.
To test this statement, vector addition and scalar multiplication have to be defined, we take them to be the obvious choices:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_m+y_m \end{bmatrix} \quad \text{and} \quad r\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} rx_1 \\ \vdots \\ rx_m \end{bmatrix}.$$

All of the vector space axioms are easy to check.

In fact, most (but not all) of the axioms follow because they are standard properties of column vectors.

The only properties which really need checking are:

- (A1) If $x, y \in \text{Null}(A)$ then $x + y \in \text{Null}(A)$.
- (A5) If $x \in \text{Null}(A)$ then $-x \in \text{Null}(A)$.
- (S1) If $x \in \text{Null}(A)$ and $r \in \mathbb{R}$ then $rx \in \text{Null}(A)$.

Examples of vector spaces–II

- The real number line: $V = \mathbb{R}$.
- The complex numbers: $V = \mathbb{C} = \mathbb{R} + i\mathbb{R}$.
- The Cartesian plane: $V = \mathbb{R}^2$.
- Three dimensional space:
  $V = \mathbb{R}^3 = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$.
- Column vectors of length m: $V = \mathbb{R}^m$
- The set of $n \times m$ matrices: $V = \mathbf{Mat}_{n,m}(\mathbb{R})$.
- The space of functions: $\mathbb{F} = \{f : \mathbb{R} \longrightarrow \mathbb{R}\}$.
- The space of polynomials of degree at most n:
  $\mathbb{P}_n =$
  $\{\, a_n x^n + \cdots + a_1 x + a_0 : a_n, \ldots, a_1, a_0 \in \mathbb{R} \,\}$.
- The space of all polynomials: (infinite dimensional)
  $\mathbb{P} = \mathbb{P}_0 \bigcup \mathbb{P}_1 \bigcup \mathbb{P}_2 \bigcup \ldots$.

In order to check that these are vector spaces we need to check all 10 of the vector space axioms!

Later we will give a formal definition of the dimension of a vector space.

Definition A Euclidean Vector Space is a vector space which has finite dimension and is equipped with an Inner Product $\langle ., . \rangle$. The Axioms for an Inner Product are P1 - P4 listed below (ref Poole Ch7):

P1 (commutativity) $\langle x, y \rangle = \langle y, x \rangle$.

P2 (distributivity) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$.

P3 (scalar multiplication)
$\langle rx, y \rangle = r\langle x, y \rangle = \langle x, ry \rangle$.

P4 (positivity) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

For $V = \mathbb{R}^n$ we use $\langle ., . \rangle$ to denote
the *Standard Inner Product* on $\mathbb{R}^n$, so that
$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + ... + x_n y_n$

The *Length* of the vector x is defined to be
$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{(x_1 x_1 + x_2 x_2 + x_3 x_3 + ... + x_n x_n)}$

The angle between two non-zero vectors x and y is the unique $0 \leq \theta \leq \pi$ such that $cos(\theta) = \langle x, y \rangle / (\|x\| \|y\|)$

Definition Two vectors x and y are said to be *orthogonal* if $\langle x, y \rangle = 0$.