Exercises for Algebra II | Dr. Geordie Williamson
Bonn, WS 2012/13 | Hanno Becker

Sheet 3

Hand in on Thursday, 1st November, prior to the lecture.

Exercise 1

Let $\mathfrak{g}$ be a nilpotent Lie algebra. Show that the Killing form on $\mathfrak{g}$ is identically zero.
(4 points)

Exercise 2

Let κ denote the Killing form on $\mathfrak{gl}_n(\mathbb{C})$ and let $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ denote the subspaces of diagonal, strictly upper triangular and strictly lower triangular matrices, respectively.

(a) Show that $\mathfrak{h}$ is orthogonal to $\mathfrak{n}_+$ and $\mathfrak{n}_-$ with respect to κ, and that the resriction of κ to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ is non-degenerate. (Hint: It might help to give a formula for κ in terms of matrix units.) (1 point)

(b) Calculate $\mathfrak{n}_+^\perp$. (1 point)

(c) Describe the radical of the restriction of κ to $\mathfrak{h}$, and conclude that the restriction of κ to $\mathfrak{sl}_n(\mathbb{C})$ is non-degenerate. (We have seen on sheet 2 that $\mathfrak{sl}_n(\mathbb{C})$ is simple. Hence part c) also follows from the Cartan-Killing criterion.) (2 points)

Exercise 3

Prove Schur's lemma: if V is a simple, finite-dimensional $\mathfrak{g}$-module over an algebraically closed field and $f \in \mathrm{End}_{\mathfrak{g}}(V)$ is an endomorphism of V commuting with the $\mathfrak{g}$-action, then f is a scalar. (Hint: Consider the eigenspaces of f.) (4 points)

Exercise 4

Prove that abstract and classical Jordan decompositions agree in the following cases:

(a) $\mathfrak{sl}_n(\mathbb{C}) \subseteq \mathfrak{gl}_n(\mathbb{C})$

(b) $\mathfrak{sp}_{2n}(\mathbb{C}) \subseteq \mathfrak{gl}_{2n}(\mathbb{C})$

(c) $\mathfrak{so}_n(\mathbb{C}) \subseteq \mathfrak{gl}_n(\mathbb{C})$

Here, $\mathfrak{sp}_{2n}(\mathbb{C}) := \{A \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid \omega(Av, w) = -\omega(v, Aw)\}$ for ω some non-degenerate, skew-symmetric form on $\mathbb{C}^{2n}$, and $\mathfrak{so}_n(\mathbb{C}) := \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid \langle Av, w\rangle = -\langle v, Aw\rangle\}$ for $\langle -, -\rangle$ some non-degenerate symmetric form on $\mathbb{C}^n$. (4 points)
Remark: In fact, *any* semisimple Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ is automatically closed under the formation of classical Jordan components.