Exercises for Algebra II
Bonn, WS 2012/13

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Sheet 12

Hand in on Thursday, 24th January, prior to the lecture.

Exercise 1

Let $G \subseteq \mathrm{GL}(V)$ be a linear Lie group. Recall that

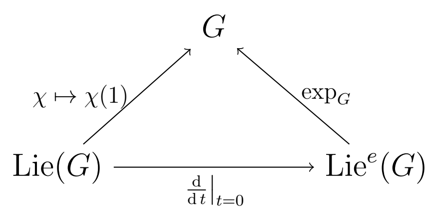
$$\mathrm{Lie}(G) := \{\chi : (\mathbb{R}, +) \to G \text{ continuous group homomorphism}\}$$

is the set of 1-parameter subgroups of G, and

$$\mathrm{Lie}^e(G) := \{X \in \mathrm{End}(V) \mid \exp(\mathbb{R} \cdot X) \subseteq G\}.$$

In the lecture, we identified $\mathrm{Lie}(G)$ with $\mathrm{Lie}^e(G)$ via $\chi \mapsto \frac{\mathrm{d}}{\mathrm{d}\,t}\big|_{t=0} \chi$ and $X \mapsto (t \mapsto \exp(tX))$; in particular, $\mathrm{Lie}(G)$ inherits the structure of an $\mathbb{R}$-vector space from $\mathrm{Lie}^e(G)$.

(a) Check that the diagram



is commutative. (1 point)

(b) Give an explicit description of the $\mathbb{R}$-vector space structure on $\mathrm{Lie}(G)$. (1 point)

Exercise 2

Show that the exponential map $\exp : \mathfrak{gl}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R})$ is not surjective. (2 points)

Exercise 3

Let V be an n-dimensional real vector space and fix a complete flag $\{0\} = V_0 \subset V_1 \subset ... \subset V_{n-1} \subset V_n = V$ in V, i.e. $\dim_{\mathbb{R}} V_i = i$. Further, as in the lecture, put $\mathcal{B} := \{\varphi \in \mathrm{GL}(V) \mid \varphi(V_i) \subseteq V_i\}$ and $\mathcal{N} := \{\varphi \in \mathrm{GL}(V) \mid (\varphi - \mathrm{id})(V_i) \subseteq V_{i-1}\}$.

(a) Show that $\mathrm{Lie}^e(\mathcal{B}) = \mathfrak{b} := \{\psi \in \mathrm{End}(V) \mid \psi(V_i) \subseteq V_i\}$. (*Hint: Use Exercise 2 from Sheet 11.*) (2 points)

(b) Show that $\mathrm{Lie}^e(\mathcal{N}) = \mathfrak{n} := \{\psi \in \mathrm{End}(V) \mid \psi(V_i) \subseteq V_{i-1}\}$. (2 points)

(c) Show that the exponential function of $\mathcal{N}$ is a diffeomorphism $\mathfrak{n} \to \mathcal{N}$. (2 points)

Exercise 4

(a) Show that $\mathrm{SO}(n;\mathbb{R})$ is path-connected. (*Hint: Remember the normal form for orthogonal matrices.*) (3 points)

(b) Show that $\mathrm{O}(n;\mathbb{R})$ has $\mathrm{SO}(n;\mathbb{R})$ and $\mathrm{O}(n;\mathbb{R})_- := \{X \in \mathrm{O}(n;\mathbb{R}) \mid \det(X) = -1\}$ as its path-components. (1 point)