## Tilting modules for reductive groups and the Hecke category



Basic goal: Understand the mod p ("i.e. "modular") representation theory of finite and algebraic groups.

Why: Suspect that there is a beautiful, rich and useful theory there.

History is helpful to understand where we are going:

The semi-simple world:

Frobenius (1896): discovery of character table of a finite group.

1897: discovers that ( gives the traces of irreducible representations:

$$\rho: G \longrightarrow Gl_n(\mathbb{C}).$$

Mashke, a few months later: proves that any ~~sub~~ finite subgroup

$G \subset Gl_n(\mathbb{C})$ acts semi-simply.

i.e. $\mathrm{Rep}_{\mathbb{C}} G$ is semi-simple.

(Might not have known about Frobenius' work.)

Frobenius 1900: determines the character table of the symmetric group.

Schur's thesis 1901: determination of the irreducible polynomial representations of $Gl_n(\mathbb{C})$ through a dictionary to the symmetric group. ("Schur-Weyl duality".)

Weyl 1925–1940: representation theory of compact Lie groups.

- $\mathrm{Rep}_{f.d.} G$ is semi-simple
- Peter-Weyl theorem etc.

(integration over a compact group etc)

Remark: Weyl's use of geometric methods was very scary at the time.
Several believed would never have an algebraic proof.

The non semi-simple world

Brauer 1935–[redacted]: modular representations of ~~groups~~ finite groups.

decomposition numbers, local / global correspondences.

Harish-Chandra 1950–: representations of semi-simple Lie groups.

Koszul philosophy (Beilinson-Ginzburg-Soergel 1996)

Def: A Koszul algebra ~~ring~~ is a positively graded ring $A = \bigoplus_{j \geq 0} A^j$

such that ① $A^0$ is semi-simple;

② $A^0 = A/A^{>0}$ has a graded projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \twoheadrightarrow A^0$$

such that $P_i = AP_i^i$ ($P_i$ generated in degree $i$).

Example: ① $k[X] \xrightarrow{\cdot X} k[X] \twoheadrightarrow k$, shows that $k[X]$ w/ $\deg X = 1$ is Koszul.

② $\xrightarrow{\cdot \varepsilon} k[\varepsilon] \xrightarrow{\cdot \varepsilon} k[\varepsilon] \xrightarrow{\cdot \varepsilon} k[\varepsilon] \longrightarrow k$ shows that $k[\varepsilon] = k[X]/(X^2)$ w/ $\deg \varepsilon = 1$ is Koszul.

③ $k[X]/(X^3)$ is not Koszul.

Slogan: (BGS) Koszul rings are "graded semi-simple".

Since BGS, Koszul duality phenomena have been discovered throughout rep theory:

③

Suppose all $A^i$ are f.g. $A^0$-modules. Then

$E(A) := \mathrm{Ext}^\bullet_A(A_0, A_0)$ is again a Koszul ring, "Koszul dual".

$A^! := E(A)^{opp}$ ⇝ equivalence of derived categories:

$\mathcal{K}: \quad D^b(A\text{-grmod}^{f.g}) \xrightarrow{\sim} D^b(A^!\text{-grmod}^{f.g})$.

(Roughly speaking, this equivalence is because "$E(A)$ is formal").

<u>Examples:</u>

① Category $\mathcal{O}$ for a complex semi-simple Lie algebra (Soergel, BGS).

② admissible representations of real semi-simple Lie groups:

Soergel's conjecture: Koszul duality "is" Langlands duality.

The moral is that although these categories are not semi-simple, they fail to be semi-simple in a simple way.

More important for this course will be two further examples:

$G$ split reductive group / $\mathbb{Z}$, $\mathfrak{g}$ its Lie algebra.

① $\mathrm{Rep}_0\, U_\varepsilon(\mathfrak{g})$ principal block of representations of Lusztig form of quantum group at an $\ell^{th}$ root of unity.

② $\mathrm{Rep}$ all blocks of restricted representations of $\mathfrak{g}_{\mathbb{F}_p}$ for $p >> 0$.

Both categories are "approximations" to $\mathrm{Rep}\, G_{\mathbb{F}_p}$.

<u>Basic Q:</u> What is the structure of $\mathrm{Rep}\, G_{\mathbb{F}_p}$.

Frobenius twist $\Rightarrow$ cannot be Koszul.

(this will be explained later)

Suspect:

$$\begin{array}{ccccccc} \text{semi-simple} & \rightleftarrows & \text{Koszul} & \rightleftarrows & \cdots & \rightleftarrows \;\cdots & \mathrm{Rep}\, G_{\mathbb{F}_p} \\ \text{generation } 0 & & \text{generation } 1 & & \text{generation } 2 & & \text{generation } \infty \end{array}$$

<u>Goal of these lectures:</u> Try to get some understanding of by careful analysis of $G = SL_2, SL_3$. Even here we don't really understand what is going on!

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$\Bbbk$ a field of char $p$.

$G \supset B \supset T$ split reductive group / $\mathbb{F}_p$

$X^*(T)$ character lattice

$\cup$

$\Phi$ roots

$\cup$

$\Phi_+$ positive roots s.t. $-\Phi_+$ are the roots occurring in $\mathrm{Lie}\, B$.

$\cup$

$\Delta$ simple roots.

$(W_f, S_f)$ "finite" Weyl group, $W = \mathbb{Z}\Phi \rtimes W_f$ affine Weyl group.

<u>Rmk:</u> $W$ is usually what people would call affine Weyl group of <u>dual</u> group.

Want to study: $\mathrm{Rep}\, G =$ <u>algebraic</u> representations of $G$ $=$ if $\Bbbk$ is infinite: reps whose matrix coefficients are in $\Bbbk($

$=$ reps of group scheme $G =$ right $\Bbbk[G]$-comodules.

$\mathcal{X}_+ \subset \mathcal{X}(T)$ dominant weights.

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$\lambda \in \mathcal{X}_+$

$$\rightsquigarrow \nabla_\lambda := H^0(G/B, \mathcal{L}_\lambda) = \operatorname{ind}_B^G \Bbbk_\lambda$$

line bundle on $G/B$ assoc. to $\lambda$.

$$\Delta_\lambda := H^0(G/B, \mathcal{L}_{-w_0\lambda})^*, \quad w_0 \in W \text{ longest elt.}$$

E.g. $G = Sl_2$, $\mathcal{X} = \mathbb{Z}$, $\nabla_n = H^0(\mathbb{P}^1, \mathcal{O}(n))$ = homogeneous polys of degree $n$.

$$\Delta_n = \nabla_n^* .$$

Thm (Chevalley) ① $\operatorname{soc} \nabla_\lambda = \operatorname{head} \Delta_\lambda =: L_\lambda$ is simple.

② $\{\text{simple } G\text{-modules}\}/\cong \;\xleftrightarrow{\sim}\; \mathcal{X}_+$

$L_\lambda \longleftarrow\!\shortmid \lambda$

Tilting modules: Fundamental vanishing:

$$\operatorname{Ext}^i(\Delta_\lambda, \nabla_\mu) = \begin{cases} \Bbbk & \text{if } \lambda = \mu, i = 0; \\ 0 & \text{otherwise.} \end{cases}$$

(Rmk: Proof not difficult via $B$-cohomology.)

Hence $\Delta_\lambda$, $\nabla_\mu$ form a "dual basis" in $D^b(\operatorname{Rep} G)$.

Def: $T$ is tilting if $T$ admits a filtrations w/ sections $\Delta_\lambda$ and a " " " $\nabla_\mu$.

Lemma: $T, T'$ tilting $\Rightarrow$ $\operatorname{Ext}^i(T, T') = 0$ for $i > 0$.

Thm (Donkin) $\{\text{ind. tilting modules}\}/\cong \;\xleftrightarrow{\sim}\; \mathcal{X}_+$

$$T_\lambda \longleftarrow\!\shortmid \lambda$$

$$\begin{pmatrix} \Delta_\mu \\ \mu < \lambda \\ \Delta_\lambda \end{pmatrix} \quad " \quad \begin{pmatrix} \nabla_\lambda \\ \nabla_\mu, \mu < \lambda \end{pmatrix}$$

<u>Example:</u> For $G = SL_2$, $\Delta_0, \Delta_1, \dots, \Delta_{p-1}$ are simple, and isomorphic to $\nabla_0, \nabla_1, \dots, \nabla_{p-1}$.

Hence $T_i = \Delta_i = \nabla_i = L_i$ for $0 \le i \le p-1$.

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For $i = p$ we have an exact sequence

$$L_p = kX^p \oplus kY^p \hookrightarrow \nabla_p \twoheadrightarrow L_{p-2}. \qquad \begin{array}{c} L_{p-2} \\ \hline L_p \end{array}$$

$$L_{p-2} \hookrightarrow \Delta_p \twoheadrightarrow L_p \qquad \begin{array}{c} L_p \\ \hline L_{p-2} \end{array}$$

$$\begin{array}{c} \Delta_{p-2} \\ \Delta_p \end{array} = \begin{array}{c} \nabla_p \\ \nabla_{p-2} \end{array}$$

and $T_p$ is a kind of "libre product":

$$\begin{array}{c} L_{p-2} \\ \hline L_p \\ \hline L_{p-2} \end{array} = \begin{array}{c} \Delta_p \\ \Delta_{p-2} \\ \nabla_p \end{array}$$

<u>Tensor product thms:</u>

<u>Thm</u> (Donkin, Mathieu) $T, T'$ tilting $\Rightarrow$ $T \otimes T'$ tilting.

<u>Rmk:</u> best proof nowadays is via canonical basis.

<u>Example:</u> For $Sl_2$, $V \otimes \Delta_{p-1} = T_p$.
$$V \otimes \Delta_{p-1} = V \otimes T_{p-1}.$$

Fundamentally new feature in char $p$: Frobenius twist.

$X_{<p} = \{\lambda \in X_+ \mid \langle \alpha^\vee, \lambda \rangle \le p-1 \ \forall \ \alpha \in \Delta\}$.

"fundamental box".

For simple

<u>Thm</u> (Steinberg) If $\lambda \in X_{<p}$, $\gamma \in X_+$ then $L_\lambda \otimes L_\gamma^{Fr}$ is semi-simple.

<u>Rmk:</u> If $G$ is simply-connected, this reduces study of Rep $L_\lambda$ to finite set.

<u>Thm</u> (Donkin) $(p \geqslant 2h-2)$, $\lambda \in (p-1)\rho + X_{<p}$, $\gamma \in X_+$.

$$T_{\lambda + p\mu} = T_\lambda \otimes T_\mu^{Fr}$$

<u>Rmk:</u> This gives a complete answer only for $Sl_2$.

In general the critical steps are missing.

## Why tilting modules?

① Combinatorics: $S^p_\lambda := \mathrm{ch}\, T_\lambda \in \mathbb{Z}[X]^W$.

Then $S^p_\lambda \in \bigoplus \mathbb{Z}_{\geq 0}$ ( ~~ch~~ Weyl Character$_\mu$ ),

and $S^p_\lambda S^p_\mu = \sum m^\nu_{\lambda\mu} S^p_\nu$ , with $m^\nu_{\lambda\mu} \geq 0$.

② Connection to symmetric group decomposition numbers:

$\lambda$ ~~p-regular~~, $\mu$ p-regular partitions:

$$d_{\mu,\lambda} \overset{\text{Erdmann, Donkin}}{=} (T_\mu : \Delta_\lambda) \quad \text{for } GL_n.$$

$$[\underbrace{S_\lambda}_{\text{Specht module}} \otimes_{\mathbb{Z}} \mathbb{F}_p : \underbrace{D_\mu}_{\text{simple module } \mu \text{ p-regular}}]$$

Hence tilting modules for $GL_n$ determine decomposition numbers for partitions w/ $\leq n$ parts.

More generally: $\mathrm{End}_{GL(V)} (\underbrace{V^{\otimes n}}_{\text{tilting module}}) \twoheadleftarrow kS_n$ , hence we can hope to understand blocks of $S_n$ via tilting modules.

③ Because $\mathrm{Ext}^i(T,T') = 0$ for $i>0$ the obvious functor

$$K^b(\mathrm{Tilt}) \xrightarrow{\sim} D^b(\mathrm{Rep})$$

+ finite homological dimension
+ existence of enough tilting objects

is an equivalence. Thus all the homological algebra of Rep is already seen in Tilt.

④ Donkin's original motivation was from invariant theory:

describe the invariants for $GL_n \circlearrowright \mathrm{Mat}_n \times \mathrm{Mat}_n \times \ldots \times \mathrm{Mat}_n$

along by simultaneous conjugation.

Some more combinatorics:



$W \circlearrowright_{\cdot p} \mathcal{X}$ "p-dilated dot action".

$W_f \ni x \cdot_p \lambda := x(\lambda+\rho) - \rho$. $\quad \mathcal{X} \ni t_\mu \cdot \lambda := \lambda + p\mu$.

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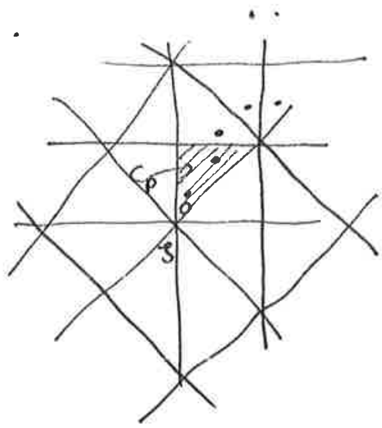
Fundamental domain for $W \circlearrowright_{\cdot p} \mathcal{X}_{\mathbb{R}}$ is given by

$$C_p := \{ \lambda \in \mathcal{X}_{\mathbb{R}} \mid 0 \le \langle \alpha^\vee, \lambda+\rho \rangle \le p \quad \forall \alpha \in \Phi_+ \}.$$

For $\bar{\mu} \in C_p$ let:

$$\mathrm{Rep}_{\bar{\mu}} := \langle L_\lambda \mid \lambda \in W \cdot_p \bar{\mu} \rangle.$$




Linkage principle:

$$\mathrm{Rep} = \bigoplus_{\bar{\mu} \in C_p \cap \mathcal{X}_+} \mathrm{Rep}_{\bar{\mu}}.$$

Assume $p \geq h$, then $0 \in \operatorname{int} C_p$.
$\mathrm{Rep}_0$ = principal block.

Many questions concerning Rep can be reduced to $\mathrm{Rep}_0$.

$^f W$ = minimal coset representatives for $W_f \backslash W$.

Then $\quad {}^f W \xleftrightarrow{\sim} (W \cdot_p 0) \cap \mathcal{X}_+$

$\quad x \longmapsto x \cdot_p 0$.

We use this to relabel $\quad L_x := L_{x \cdot 0}, \quad \Delta_x := \Delta_{x \cdot 0}, \quad T_x := T_{x \cdot 0}$ etc.

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(5) By results of Andersen, knowledge of $\mathrm{ch}\, T_\lambda \Rightarrow \mathrm{ch}\, L_\lambda$ for $p \geq 2h-2$.

For a fixed group, it appears much more complicated to calculate $\mathrm{ch}\, T_\lambda$ than $\mathrm{ch}\, L_\lambda$.