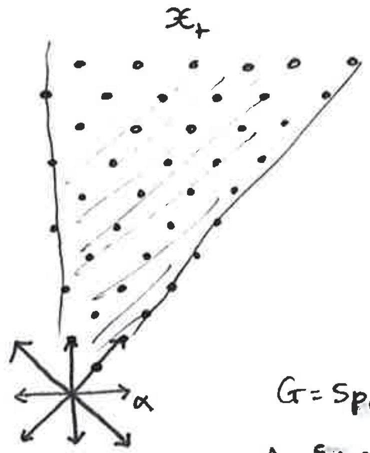
① Last time:

G split reductive / k.
$\cup$
B Borel subgroup
$\cup$
T maximal torus.

Starting from now:

$\operatorname{Rep} G :=$ finite-dimensional algebraic representations of G. $\subset$ all reps $=$ ind Rep G.

$\Phi \supset \Phi_+ \supset \Delta$
$\cap$
$\mathfrak{X} := \mathfrak{X}(T)$ character lattice
$\cup$
$\mathfrak{X}_+$ dominant weights



$G = Sp_4$
$\Delta = \{\alpha, \beta\}$.

$\lambda \in \mathfrak{X}_+ \rightsquigarrow \Delta_\lambda, \nabla_\lambda, L_\lambda, T_\lambda$ module.
(standard, costandard, simple, ind. tilting)

Characters:

Basic fact/exercise: any T-module is completely reducible, hence $\cong \bigoplus_{\lambda \in \mathfrak{X}} V_\lambda$.

Given $V \in \operatorname{Rep} G \rightsquigarrow \operatorname{ch} V = \sum (\dim V_\lambda) e^\lambda \in (\mathbb{Z}[\mathfrak{X}])^{W_f}$.

Kempf vanishing thm $\Rightarrow$ $\operatorname{ch} \Delta_\lambda = \operatorname{ch} \nabla_\lambda =$ Weyl character (λ).

$$= \sum \frac{(-1)^{\ell(x)} e^{x \cdot \lambda}}{\prod_{\alpha \in \Phi_+} (1 - e^{-\alpha})}.$$

Frobenius twist: $\Rightarrow G$ arises via extension of scalars from $G_{\mathbb{Z}}$ (Chevalley scheme). (classification of split reductive alg. groups)

In particular, G has an $\mathbb{F}_p$-structure $\rightsquigarrow$ $\operatorname{Fr}: G \to G$ Frobenius.

Eg. for sl_2: $\operatorname{Fr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$.

Very important for two reasons:

① $G^{(\operatorname{Fr})^m} = G(\mathbb{F}_{p^m})$ finite group of Lie type.
(Lie groups don't contain many interesting finite groups, whereas algebraic groups in char. p do!)

② Given a representation V of G, get V^{Fr} "Frobenius twist" via

$$G \xrightarrow{Fr} G \longrightarrow GL(V).$$

<u>Exercise</u>: If $ch(V) = \sum m_\lambda e^{\lambda}$ then $ch(V^{Fr}) = \sum m_\lambda e^{p\lambda}$.

<u>Exercise</u>: If V is simple then so is V^{Fr}.

Thus representations in char p can contain many "holes".

(Harmless for study of Rep G.)

Assume G is semi-simple, simply connected. Define $X_{<\ell} = \{\lambda \mid 0 \le \langle \alpha^\vee, \lambda\rangle \le \ell-1, \forall \alpha\in\Delta\}$

$$= \{\sum_{\alpha\in\Delta} \lambda_\alpha \varpi_\alpha \mid 0 \le \lambda_\alpha \le \ell-1\}$$

"fundamental box".

<u>Steinberg Tensor product theorem</u>: Fix $\lambda \in X_{<p}$.

If L is simple and $\lambda \in X_{<p}$, then $L^{Fr} \otimes L_\lambda$ is too.

<u>Corollary</u>: If $\lambda = \sum_{i=0}^{m} \lambda_i p^i$ with $\lambda_i \in X_{<p}$, then

$$L_\lambda \cong L_{\lambda_0} \otimes L_{\lambda_1}^{Fr} \otimes \ldots \otimes L_{\lambda_m}^{(Fr)^m}.$$

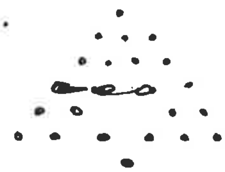
<u>Rmk</u>: Hence description of simple G-modules is a finite problem.

<u>Eg. for SL_2</u>: All simple modules are obtained by tensor products of twists of $L_0, \ldots, L_{p-1}$, i.e. homogeneous polys of degs $0, 1, \ldots, p-1$.

<u>Exercise</u>:

$$ch\, L_m = \sum_{\substack{0 \le i \le m \text{ s.t.} \\ \binom{m}{i} \not\equiv 0 \bmod p}} e^{m-2i}.$$

"Sierpinski triangle".



<u>Steinberg restriction thm</u>: ① If $\lambda \in X_{<p^m}$ then $\mathrm{Res}_G^{G(\mathbb{F}_{p^m})} L_\lambda$ is simple.

② $\mathrm{Res}_G^{G(\mathbb{F}_{p^m})}$ provides a bijection:

$$\left\{ \begin{matrix} \text{simple} \\ \Bbbk G(\mathbb{F}_{p^m}) \\ \text{modules} \end{matrix} \right\} / \cong \;\longleftrightarrow\; X_{<p^m}.$$

<u>Rmk</u>: Many aspects of Rep $\Bbbk G(\mathbb{F}_{p^m})$ remain mysterious, even where we understand Rep G.

(3) What about tilting modules?

tilting characters = characters of indecomposable tilting modules.

~~Sup~~ Assume $p \geq 2h-2$ (should be unnecessary).

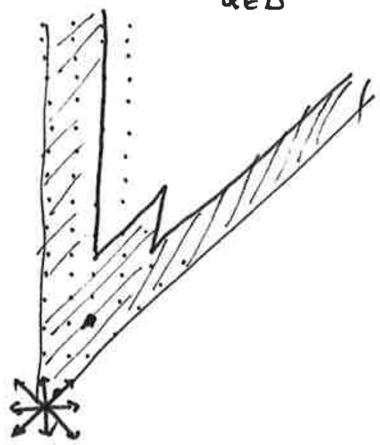
<u>Donkin:</u> Suppose $\lambda \in (p-1)\rho + X_{<p}$.

If T is ind. tilting then so is $T^{Fr} \otimes T_\lambda$.

<u>Rmk:</u> This is only a partial reduction, and does not reduce to a finite problem. We need to know tilting characters for the set of wts:

$$(p-1)\rho + X_{<p} \quad \cup \quad \left\{ \sum_{\alpha\in\Delta} \lambda_\alpha \alpha \;\middle|\; \begin{array}{c}\text{for some } \alpha, \\ \lambda_\alpha \le p-1\end{array} \right\}.$$

"wall cases".



<u>Interesting Question:</u> what modules for $\Bbbk G(\mathbb{F}_q)$ does one get via restrictions of tilting modules?

<u>p-dilated dot action:</u>

$W_f \curvearrowright X$ via dot action: $x \cdot \lambda := x(\lambda+\rho)-\rho$.

Fundamental in mod p rep theory is p-dilated dot action:

$$x \cdot_p \lambda := x \cdot \lambda \quad \text{if } x \in W_f.$$

$$t_\mu \cdot_p \lambda := p\mu + \lambda \quad \text{if } \mu \in \mathbb{Z}\Phi. \quad (t_\mu \text{ denotes translation}).$$

$W \curvearrowright_{\cdot_p} X$ = group generated by affine reflections in the hyperplanes

$H_{\alpha,m} := \{ \langle \alpha^\vee, \lambda+\rho \rangle = pm \}$ = if $p \geq h$: hyperplanes where Weyl character formula is divisible by p.

Linkage principle: $\mathrm{Ext}^i(L_\lambda, L_\mu) \neq 0 \implies \lambda \in W_p^{\bullet}\mu$.

Hence: $\mathrm{Rep}\,G \overset{(*)}{=} \bigoplus_{\bar\mu \in \mathfrak{X}/(W_p^\bullet)} \mathrm{Rep}_{\bar\mu}$ $\qquad \mathrm{Rep}_{\bar\mu} = \langle L_\mu \mid \mu \in \bar\mu \cap \mathfrak{X}_+ \rangle$.

Rmk: A fundamental domain for $(W_p^\bullet)$-action is $C_p := \{\lambda \in \mathfrak{X}_+ \mid 0 \leq \langle \alpha^\vee, \lambda+\rho \rangle \leq p \ \forall \alpha \in \Phi_+\}$ "fundamental p-alcove".

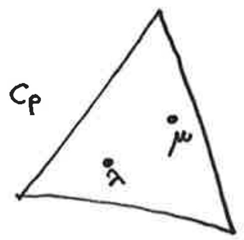
Consider an exact functor $F : \mathrm{Rep}\,G \to \mathrm{Rep}\,G$.

via. (*) we can view it as a matrix of functors $(F_{\bar\lambda,\bar\mu})_{\bar\lambda,\bar\mu \in \mathfrak{X}/(W_p)}$

This is a very useful pt of view!

Translation Functors:

$\mathrm{Rep}_\lambda \dashrightarrow \mathrm{Rep}_\mu$ How to move between these blocks in a universal way?



Let V be a G-module with extremal wt $\mu-\lambda$.
V^* extremal wt $\lambda-\mu$.

$\Theta_{\lambda\to\mu} : \mathrm{Rep}_\lambda \hookrightarrow \mathrm{Rep} \xrightarrow{(\ )\otimes V} \mathrm{Rep} \to \mathrm{Rep}_\mu$ "matrix coefficient of $(-)\otimes V$".

$\Theta^*_{\lambda\to\mu} : \mathrm{Rep}_\mu \hookrightarrow \mathrm{Rep} \xrightarrow{(\ )\otimes V^*} \mathrm{Rep} \to \mathrm{Rep}_\lambda$

Lemma ⓪ $\Theta_{\lambda\to\mu}$ is well-defined up to isomorphism (doesn't depend on V).

① $\Theta_{\lambda\to\mu}$, $\Theta^*_{\cancel{\mu\to\lambda}\ \lambda\to\mu}$ are biadjoint (obvious!)

② if $\lambda,\mu \in C_p$ have same stabilizer in ~~Rep~~ $W_p^\bullet G$, then $\Theta_{\lambda\to\mu}$, $\Theta^*_{\lambda\to\mu}$ are equivalences of abelian categories.

⑤ Extremely important for all these results is the "tensoridentity"

$$\cancel{\mathrm{ind}}\ (\mathrm{ind}_B^G M)\otimes V \xrightarrow{\sim} \mathrm{ind}_B^G\left(M\otimes \mathrm{res}_B^G V\right).$$

In particular: $\Delta_\lambda\otimes V \xrightarrow{\sim} \mathrm{ind}_B^G\left(k_\lambda\otimes \mathrm{res}_B^G V\right).$

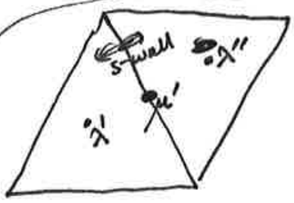
Rmk: I always find this formula hard to internalize. Easier:

$$\begin{array}{c} pt/B \\ \downarrow \pi \\ pt/G \end{array}$$

Projection formula: $\pi_*(\mathcal{F}\otimes\pi^*\mathcal{G}) = (\pi_*\mathcal{F})\otimes\mathcal{G}.$

$$\mathrm{ind}_B^G(M\otimes \mathrm{res}_B^G V) \qquad \mathrm{ind}_B^G M\otimes V.$$

Very important consequences:



$\lambda', \lambda''\in W_p\bullet\lambda$

$\mu'\in W_p\bullet\mu$

ⓐ $\Theta_{\lambda\to\mu}(\Delta_\lambda) = \begin{cases} 0 & \text{if } \mu' \text{ is not dominant.} \\ \Delta_\mu & \text{if } \mu' \text{ is dominant} \end{cases}$

If μ' is dominant, there is a short exact sequence.

ⓑ $\Delta_{\lambda''}\hookrightarrow \Theta_{\mu\to\lambda}(\Delta_\mu)\twoheadrightarrow \Delta_{\lambda'}$

Wall-crossing functors: Assume $p\geq h$ and fix weights $\mu_s\in C_p$ such that $\mathrm{Stab}_{W_p}(\mu_s)=\{s\}$ for all $s\in S$.

Define $\Theta_s : \Theta_{\mu_s\to 0}\circ\Theta_{0\to\mu_s} : \mathrm{Rep}_0\longrightarrow \mathrm{Rep}_0.$

The weights in Rep_0 are $(W_p\bullet 0)\cap\mathcal{X}_+ = \{x\bullet_p 0 \mid x\in {}^fW\}\xrightarrow{\sim}{}^fW.$

Use ${}^fW$ to relabel our elements $\Delta_x := \Delta_{x\bullet_p 0}$, $T_x := T_{x\bullet_p 0}$ etc.

⑥ ⓐ and ⓑ above show:

$$\theta_s([\Delta_x]) = \begin{cases} [\Delta_x] + [\Delta_{xs}] & \text{if } xs \in {}^f W \\ 0 & \text{if } xs \notin {}^f W. \end{cases}$$

Fundamental observation:

let sgn $= \mathbb{Z}\varepsilon$ denote the sign rep of W_f. Then

$$\mathrm{ind}\ \mathrm{sgn} \otimes_{\mathbb{Z}W_f} \mathbb{Z}W = \bigoplus_{x \in {}^f W} \mathbb{Z}(\varepsilon \otimes x)$$

and we have, for any $s \in S$:

$$\varepsilon \otimes x \cdot (1+s) = \begin{cases} \varepsilon \otimes x + \varepsilon \otimes xs & \text{if } xs \in {}^f W \\ \varepsilon \otimes x + \varepsilon \otimes s'x = 0 & \text{if } xs \notin {}^f W. \end{cases}$$

(with $s' \in S_f$)

(Fact: if $x \in {}^f W$, $xs \notin {}^f W$ then $xs = tx$ for some $t \in S_f$.)

Conclusion:

$$\begin{array}{ccc} \mathrm{sgn} \otimes_{\mathbb{Z}W_f} \mathbb{Z}W \circlearrowleft (1+s) & \xrightarrow{\sim} & [\mathrm{Rep}_0] \circlearrowleft \theta_s \\ \cup & & \cup \\ \varepsilon \otimes x & \longmapsto & [\Delta_x] \end{array}$$

isomorphism of $\mathbb{Z}W$-modules!

" Rep_0 categorifies the anti-spherical module".

Fundamental question: what can one say about the structure of translation functors, what is the monoidal category of endo-functors that they generate??

⑦ <u>Conjecture</u> (w/ S. Riche)

full Karoubian monoidal category generated by wall-crossing functors

exact endofunctors of Rep_0

$$\phi: \mathcal{H} \longrightarrow \langle \Theta_s \rangle_{s \in S} \subset$$

$$\begin{array}{ccc} \Psi & & \Psi \\ B_s & \longmapsto & \Theta_s \end{array}$$

Hecke category (categorification of $\mathbb{Z}W$)

Categorification of the fact that $\mathbb{Z}W = \langle (1+s) \rangle_{s \in S} \subset \mathrm{End}(\mathrm{sgn} \otimes \mathbb{Z}W)$.

<u>Rmk:</u> Don't expect ϕ to be essentially surjective, e.g. $\mathrm{Hom}_{\mathcal{H}}(\mathbb{1}, \mathbb{1}) \longrightarrow \mathrm{Hom}(id, id) =$ "centre" of Rep_0 is not surjective for SL_2.

<u>Philosophy:</u> (higher representation theory)

It should be fruitful to study Rep_0 as a module over $\mathcal{H}$.

~~Rmk~~

<u>Hecke category:</u> Will be brief, please see intro to "Soergel calculus" or recent ICM report "Parity sheaves and the Hecke category" for more detail.

First: Hecke algebra

$(W, S) \rightsquigarrow$ Hecke algebra: $H = \langle t_s \mid s \in S \rangle / \mathbb{Z}[v^{\pm 1}]$ subject to

$$t_s^2 = (v^{-1} - v) t_s + 1$$

$$\underbrace{t_s t_{s'} \dots}_{m_{ss'}} = \underbrace{t_{s'} t_s \dots}_{m_{ss'}} \quad \text{braid relations.}$$

For a reduced expression $\underline{w} = (s_1, \dots, s_m)$ for w set $t_w := t_{s_1} \dots t_{s_m}$.

t_w are well-defined and provide a $\mathbb{Z}[v^{\pm 1}]$-basis of H.

Kazhdan-Lusztig involution $d(t_x) = t_{x^{-1}}^{-1}$, $v \mapsto v^{-1}$.

KL polynomials

$\exists!$ basis $\{b_w\}$ s.t. $d(b_w) = b_w$ and $b_w = t_w + \sum_{y < w} h_{y,w} t_y$, $h_{y,w} \in \mathbb{Z}[v]$.

Rmk: H is also generated by $\{b_s\}_{s\in S}$ subject to relations:

$$b_s^2 = (v+v^{-1})\, b_s$$

$$b_s b_{s'} \cdots - b_{s'} b_s \cdots = \text{complicated but explicit. shorter expression of } b_s, b_{s'}\text{'s.}$$

E.g. $m_{s,s'}=2$: $b_s b_{s'} - b_{s'} b_s = 0$, $m_{s,s'}=3$

$$b_s b_{s'} b_s - b_{s'} b_s b_{s'} = b_s - b_{s'}.$$

Plan of construction:

$$\mathcal{H}_{BS} \xrightarrow{\ \text{I}\ } \mathcal{H}_{BS}^{\oplus,[\mathbb{Z}]} \xrightarrow{\ \text{II}\ } \mathcal{H}$$

$\mathcal{H}_{BS}$: monoidally generated by S (category with objects $\leftrightarrow$ expressions in S), morphisms given by planar diagrams, enriched in graded vector spaces.

I: formally add shifts and $\oplus$, take deg 0 morphisms.

II: take Karoubi envelope.

Thm:

$$[\mathcal{H}]_\oplus \xleftarrow{\ \sim\ } H$$

$$[B_s] \longmapsfrom b_s$$

split Grothendieck group,

ring $[M]\cdot[N] := [MN]$.

$\mathbb{Z}[v^{\pm 1}]$-module $v^m \cdot [M] = [M(m)]$.

Both steps I and II are formal.

See lectures: "MSRI Soergel bimodules"

"Aarhus masterclass."