Characters of tilting modules: Today: SL_2, SL_3 (conjecture w/ Lusztig)

Next week: how we arrived at this conjecture

(return to conjecture w/ Riche)

$W \supset W_f$ affine Weyl group, finite Weyl group.

$\cup \quad\quad \cup$

$S \supset S_f$ $\quad\quad {}^f W =$ minimal coset reps for $W_f \backslash W$.

$\mathrm{sgn} = \mathbb{Z}\varepsilon$

$$AS = \mathrm{sgn} \otimes_{\mathbb{Z}W_f} \mathbb{Z}W = \bigoplus_{x \in {}^f W} \mathbb{Z}\, \varepsilon \otimes x$$

Key formula:

$$\varepsilon \otimes x \cdot (1+s) = \begin{cases} \varepsilon \otimes x + \varepsilon \otimes xs & \text{if } xs \in {}^f W \\ 0 & \text{o/w.} \end{cases}$$

H Hecke algebra, free w/ basis $\{h_x \mid x \in W\}$ and multiplication:

$$\underset{W}{t_x}\underset{S}{t_s} = \begin{cases} (v^{-1}-v)\cancel{h_x} t_x + \cancel{h_{xs}} t_{xs} & \text{if } xs < x \\ \cancel{h_{xs}} t_{xs} & \text{if } xs > x. \end{cases}$$

Kazhdan-Lusztig involution $\quad d(h_x) := h_{x^{-1}}^{-1}, \quad d(v) = v^{-1}$.

Kazhdan-Lusztig basis: $\{b_x\}$

unique s.t. $\quad b_x = \cancel{\sum_{y \le x} h_{y,x}} t_x + \sum_{y<x} h_{y,x} t_y \quad h_{y,x} \in v\mathbb{Z}[v].$

$$d(b_x) = b_x.$$

sgn_v : $\quad \varepsilon \cdot h_s = -v\varepsilon$.

$$AS_v = \mathrm{sgn} \otimes_{H_f} H = \bigoplus_{x \in {}^f W} \mathbb{Z}[v^{\pm 1}]\, n_x$$

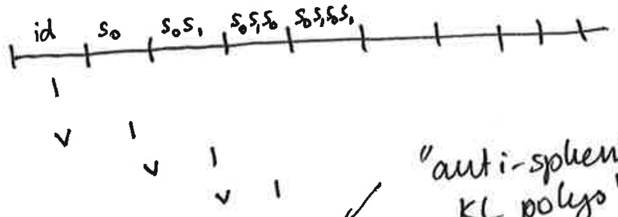
$$n_x b_s = \begin{cases} n_{xs} + v n_x & \text{if } xs > x,\ xs \in {}^f W \\ n_{xs} + v^{-1} n_x & \text{if } xs < x,\ xs \in {}^f W \\ 0 & \text{if } xs \notin {}^f W. \end{cases}$$

AS_v also has a canonical basis. $\{c_x \mid x \in {}^fW\}$.
Characterised in same way as KL basis.
One has:

$$\underset{\substack{\cap \\ AS_v}}{1 \cdot b_x} = \begin{cases} 0 & \text{if } x \notin {}^fW \\ c_x & \text{if } x \in {}^fW \end{cases}$$

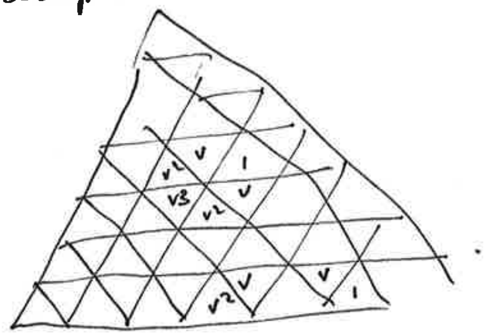
but b_x is much more computable.

SL_2: (exercise)



"anti-spherical KL polys".

SL_3:



(can be given by closed formulas).

Write $c_x = \sum h^{asp}_{y,x} n_y$.

Thm (Soergel) For $x \in {}^fW$

$$ch(T^\varepsilon_x) = \sum_{y \leq x} n^{asp}_{y,x}(1)\, ch(T^\varepsilon_y)$$

tilting module for quantum group at an ℓ^{th} root of unity w/ same root system.

Basic fact (idempotent lifting):

We can write $ch(T_x) = \sum$ positive $ch(T^\varepsilon_x)$.

Thus $ch\, T^\varepsilon_x$ is a "first approx." to $ch(T_x)$.

<u>Donkin's formula:</u> If T is indecomposable tilting and $p-1 \le m < 2p-1$ then

$$T^{Fr} \otimes T_m \text{ is also indecomposable tilting.}$$

Interesting special case: $m = p-1$, so $T_m = St = L_m$ is the <u>Steinberg module</u>.

<u>Lemma:</u> $\Delta_m^{Fr} \otimes St = \Delta_{pm+(p-1)}$.

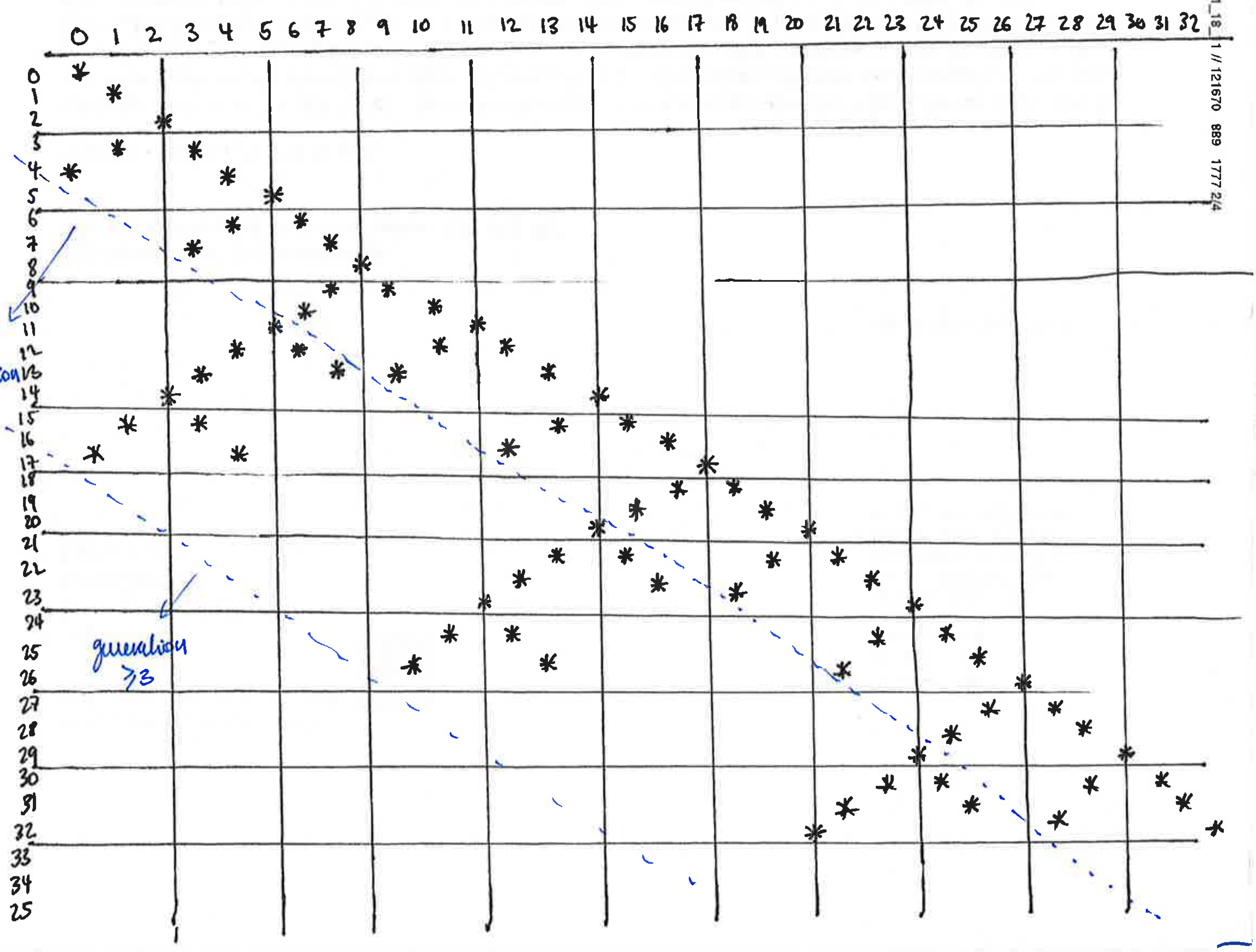
<u>Rmk:</u> On the level of chars: $LHS = \left(\frac{t^{m+1} - t^{-m-1}}{t - t^{-1}}\right)^{Fr} \left(\frac{t^{p} - t^{-p}}{t - t^{-1}}\right)$

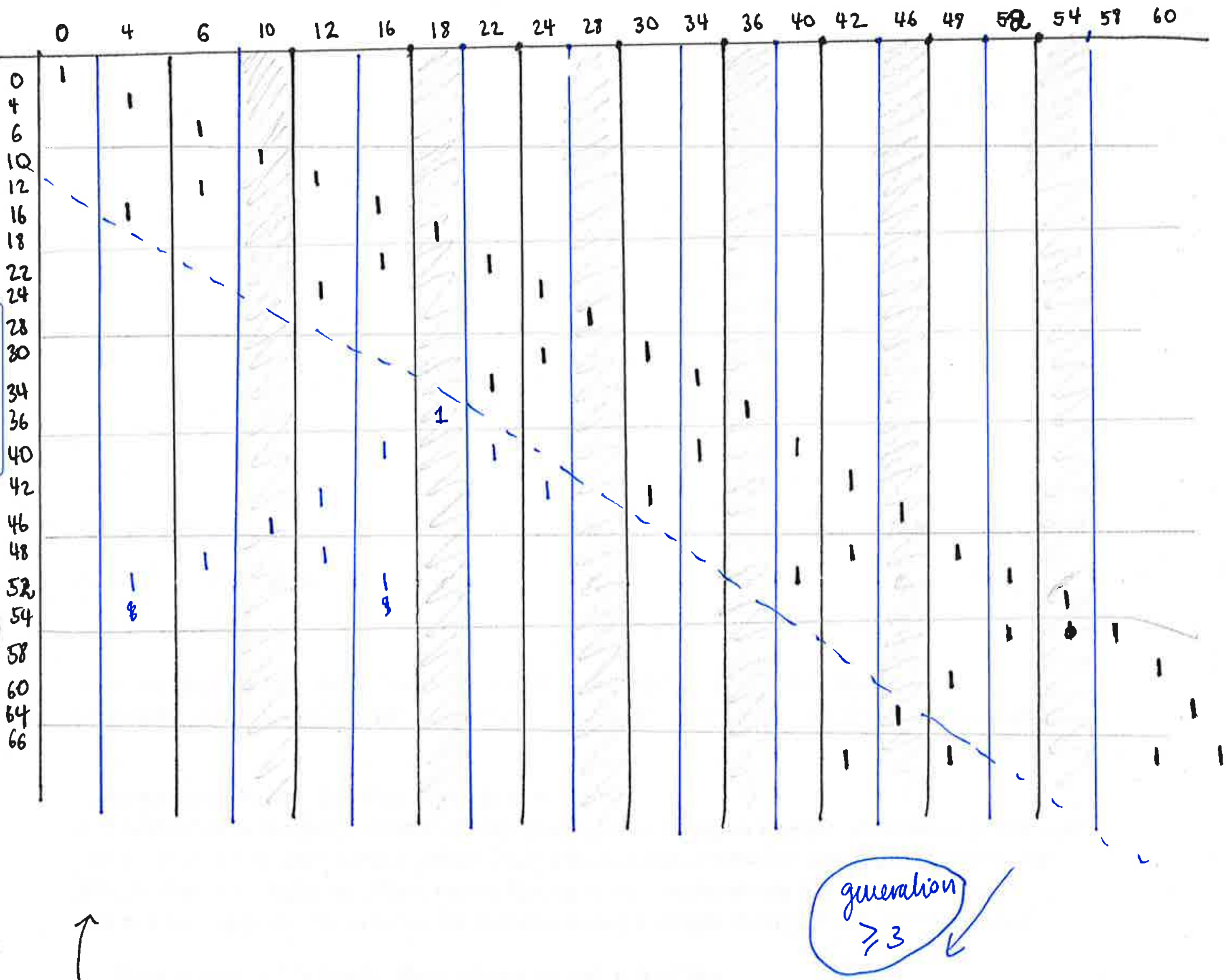
$$= \left(\frac{t^{pm+p} - t^{-pm-p}}{t^{p} - t^{-p}}\right)\left(\frac{t^{p} - t^{-p}}{t - t^{-1}}\right) = ch\, \Delta_{pm+(p-1)}.$$

Because $\otimes$ w/ Steinberg is exact, we see:

if T is indecomposable tilting with $[T] = \sum n_a [\Delta_a]$ $\Longrightarrow$ $[T^{Fr} \otimes St] = \sum n_a [T_{pa+(p-1)}]$.

We can use this to completely determine tilting chars for SL_2.





Here we take $p=3$ and depict only weights in the principal block. $(0,4,6,10,12,\dots)$

We express each: $[T_m] = \sum n_{a,m} [T_a^{\varepsilon}]$

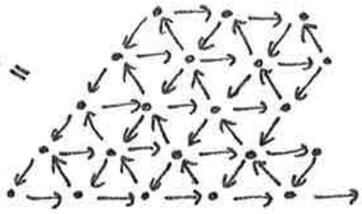
character of tilting module at p^{th} root of unity.

"Second generation adjustment matrix".

Conjecture for SL_3, generation 2:

(w/ Lusztig)

$\mathcal{X}_+$ = dominant weights = Γ =



Viewed as a graph Γ w/ $\mu \to \mu'$ if $\mu' - \mu \in W \cdot \varpi_1$

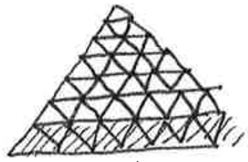
① Construct an infinite set Z of paths (sequences of elts. of $\mathcal{X}_+$) in two steps:

ⓐ "Brownian motion" ⓑ "Billiards".

② We conjecture that Z gives expressions:

$$ch(T_A^{2nd\ gen}) = \sum n_{B,A}\, ch(T_B^{1st\ gen}).$$

for



$A \in$ "wall case".

(This is tricky to explain, but not complicated.)

Rmk: This choice of wall explains assymetry in choice of ϖ_1 above.

Analogy: For SL_2, Z would consist of the paths:

Fix $\ell \geq 3$: (In applications, ℓ should be prime, but this isn't important below.)

Γ_{wall} = "ℓ-singular edges and vertices" minus wall cases. =

"Brownian motion" 1st approximation: a particle diffusing on this graph, trying not to go straight ahead!

More precisely: (sequence of wts)

Let π be a path, $(\lambda_0, \lambda_1, \dots, \lambda_k)$ two operations:

s.t. endpoint is a unique source: unique arrow $\lambda_k \to \lambda_{k+1}$.

⓪ Rest: $\pi' = (\lambda_0, \lambda_1, \dots, \lambda_k, \lambda_k)$. Charge := +3

① small step: take one step in graph to produce $\pi' = \pi \cup \{\lambda_{k+1}\}$.

Charge := +2

② giant leap: $\pi \rightsquigarrow$ take $p-1$ steps, going straight forbidden to produce one or two wts λ', λ''.

Charge := $+2\ell+1$

$\pi \rightsquigarrow$ $(\lambda_1, \dots, \lambda_k, \lambda')$ and $(\lambda_1, \dots, \lambda_k, \lambda'')$.

For each $m \geq 1$, do the following:

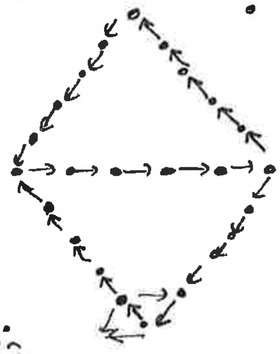
0. $\pi_0 = (m\ell\varpi_1)$.

0.5. Take $\ell-1$ small steps to produce π.

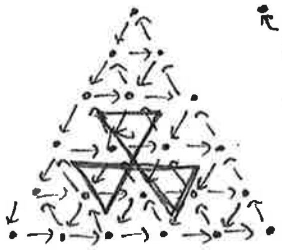
1. Take $\ell-2$ giant ~~steps~~ leaps to produce $\pi_1, \dots, \pi_j$.

2. Rest, take $\ell-1$ small steps, rest.

Now repeat ① and ② indefinitely.



First $\ell-3$ produced by ① and final produced by ② are "seeds".



"Billiards": first approximation

bounce like a billiard in the interior of each alcove.

Fix an alcove.

More precisely: Start with a pair $(\pi, CurDir)$

path; vector in $W \cdot \varpi_1$.

$\pi = (\lambda_0, \dots, \lambda_k)$. $\lambda_k \xrightarrow{CurDir} \lambda_k'$.

Produce $(\pi', CurDir')$ as follows:

1. if $\lambda_k' \in$ interior of alcove, $(\pi', CurDir') = (\varpi(\lambda_0, \dots, \lambda_k, \lambda_k'), CurDir)$

2. otherwise, $(\pi', CurDir') = ((\lambda_0, \dots, \lambda_k, \lambda_k), CurDir')$

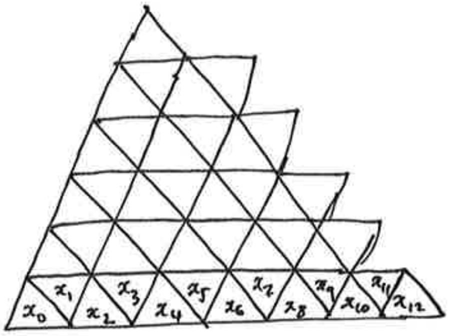
"bounce"

reflection of CurDir in wall of λ_k'.

Each seed produced by Brownian motion gives rise to 2 billiards.

$\rightsquigarrow \mathbb{Z}$.

From Z to character formulas:



${}^fW \xleftrightarrow{\sim}$ dominant alcoves

~~Any $x \in {}^fW$ can be written uniquely as~~

Open box: $B = \{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 \leq \langle \alpha^\vee, \lambda \rangle < 1$ for all $\alpha \in \Delta \}$.

For any $x \in {}^fW$ there is a unique $\lambda \in \mathcal{X}_+$ s.t. $x \cdot A_0 \subset t_\lambda(B)$.

This determines a map $\alpha : {}^fW \longrightarrow \mathcal{X}_+$.

We want to know: $ch(T_{x_i}) = \sum n_{y, x_i} \, ch(T^\varepsilon_y)$.

Instead we consider: $\xi_i = \sum n_{y, x_i} \alpha(y) \in \mathbb{Z}[\mathcal{X}_+]$.

Conjecture: ξ_i and $ch(T_{x_i})$ determine each other, for $0 \leq i \leq 2p(p+1)$.

This gives decomposition numbers for 3-row partitions with $\leq p^2(p+1)$ boxes.

Rmk: I am cheating a little, see the paper!