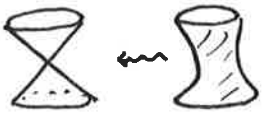
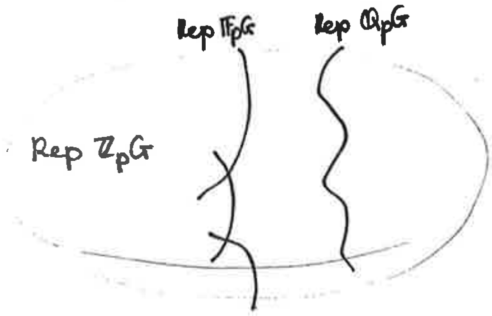
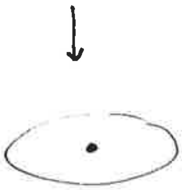
Deformation philosophy:

1 algebraic geometry:  $xy - z^2 = \lambda$ pass information between smooth and singular.

$\lambda = 0$ $\lambda \neq 0$ / Spec $\mathbb{C}[\lambda]$

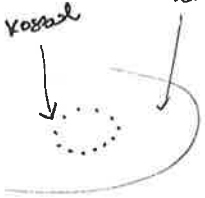
2 G finite group:



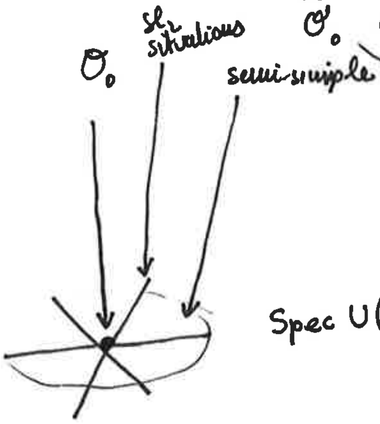

Spec $\mathbb{Z}_p$

decomposition numbers etc.

Rep H_q (Hecke alg.)
Rep ~~U_q~~$U_q(\mathfrak{g})$



3 Lie theory: $\mathfrak{g}$ a complex semi-simple Lie algebra.
$\mathfrak{h} \subset$ Cartan. $\mathcal{O}_0$ principal block of category $\mathcal{O}$
$\widetilde{\mathcal{O}}_0$ deformed category $\mathcal{O}$.



Spec $U(\mathfrak{h}) = \mathfrak{h}^*$.

Soergel philosophy: $\mathcal{O}_0$ is determined by generic and subgeneric cases.

Example: $\mathfrak{g} = sl_2(\mathbb{C})$. $\widetilde{\mathcal{O}}_0$ $\bullet \underset{f}{\overset{e}{\rightleftarrows}} \bullet$ $ef = t$.

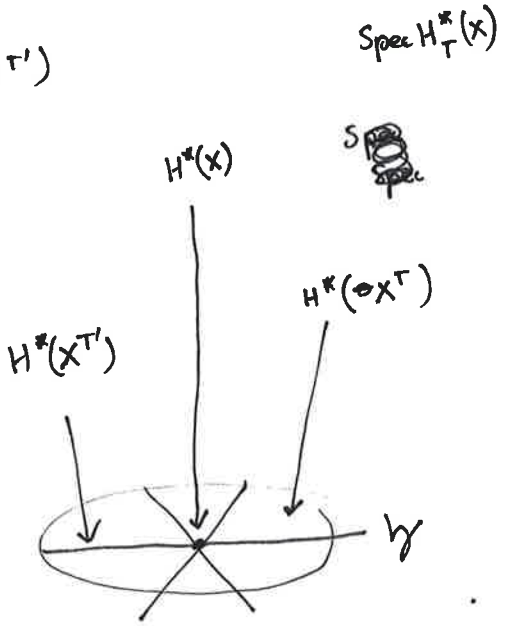
$t = 0$ $t \neq 0$
$\wr$ semi-simple.
$\mathcal{O}_0$

4 $T \curvearrowright X$ smooth projective.
tori

$H^*_T(X)$ is a module over $H^*_T(pt) = \mathcal{O}(\mathfrak{h})$.

<u>Localisation thm:</u> • $H^*_T(X) \overset{r^*}{\hookrightarrow} H^*_T(X^T) \longleftarrow H^*_T(X^{T'})$

• $\operatorname{im} r^* = \bigcap_{\substack{T' \subset T \\ \text{codim } 1}} \operatorname{im} r_{T'}$.

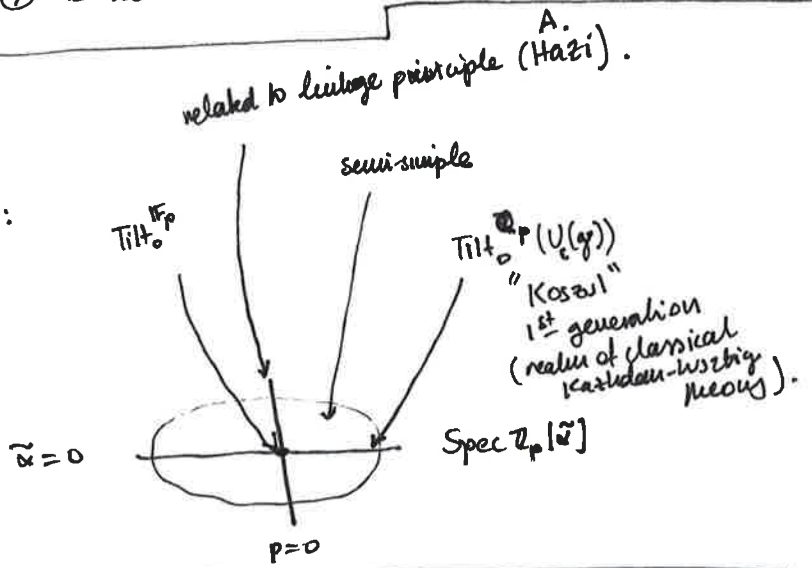


<u>Rmk:</u> This theorem is extremely useful for
1 performing calculations.

2 The similarity between 3 and 4 is no coincidence.

We want to explain a similar picture for Tilt_0:

It will take us some time to arrive at this picture.



<u>Dual geometric representation:</u> $G \rightsquigarrow W$ (affine Weyl group: $W_f \ltimes \mathbb{Z}\Phi \subset \mathrm{Aff}(X^*(T))$).
assume ~~simple~~ simple, simply connected.

$\mathfrak{h}^* = \bigoplus_{s \in S_f} \mathbb{Z}\alpha_s \oplus \mathbb{Z}\tilde{\alpha}$ affine simple root.

Define $\alpha_s^\vee \in \mathfrak{h}$ s.t. $\langle \alpha_s, \alpha_t^\vee \rangle$ = affine Cartan matrix.

$\rightsquigarrow \mathcal{H}$ Hecke category $/ \mathbb{Z}_p$.

LATER

<u>Hecke category:</u> H Hecke algebra of (W, S) / $\mathbb{Z}[v^{\pm 1}]$. $\delta_s^2 = (v^{-1} - v)\delta_s + 1$, braid rel^{ns}.

<u>Origins:</u> $G/_{\mathbb{F}_q}$ split ~~finite~~ reductive group. $B \subset G$ Borel.

Then: $$\underset{v \mapsto \frac{1}{\sqrt{|\mathbb{F}_q|}}}{H \otimes \mathbb{C}} \xrightarrow{\sim} \mathrm{Hecke}(B(\mathbb{F}_q) \subset G(\mathbb{F}_q))$$
$$= \Big(\mathrm{Fun}_{B(\mathbb{F}_q) \times B(\mathbb{F}_q)}(G(\mathbb{F}_q), \mathbb{C}), \underset{\substack{\uparrow \\ \text{convolution.}}}{*}\Big)$$

"independent of q".

To categorify H natural to consider: $\left(D^b_{B \times B}(G, \overline{\mathbb{Q}}_\ell), *\right)$

equivariant derived category.

Various technicalities (mixed sheaves, non-semi-simple Frobenius)

mean the following is better:

Take $G/\mathbb{C}$ complex reductive group, for all $s \in S$ set $P_s := \overline{BsB}$:

$$D^b_{B \times B}(G; \mathbb{Q}) \qquad \langle \underline{\mathbb{Q}}_{P_s} \mid s \in S \rangle_{*, [\mathbb{Z}], \oplus, \text{ Karoubi}} =: \mathcal{H}.$$

$\| \leftarrow$ Decomp. Thm

$\langle$ semi-simple complexes $\rangle$

For general Coxeter groups one can do this by generators and relations:

Result: $\mathcal{H} / \mathbb{Z}$ Hecke category.

$\mathcal{H}_{BS}$ monoidal cat

additive, with shift functor (n).

<u>Main Thm:</u> ① For any complete local ring or field $\Bbbk$,

self-dual indec. objects in $\mathcal{H}_\Bbbk \leftrightarrow W$

$B_w \longleftrightarrow w$

③ If $\Bbbk = \mathbb{Q}$:

$[B_w] \mapsto b_w$

(Kazhdan-Lusztig basis).

② $[\mathcal{H}_\Bbbk]_\oplus \xrightarrow{\sim} H$ Hecke algebra.

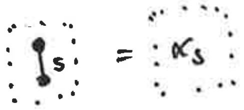
$\mathcal{H}$ is constructed in 3 steps:

1. $\mathcal{H}_{BS}$: monoidal category with:
   - objects sequences of elts. of S;
   - morphisms isotopy classes of planar diagrams with generators

$1 \quad 1 \quad -1 \quad -1 \quad \deg f \quad 0$

graded by degree

modulo local relations.

Eg: 

2. $\mathcal{H}_{\oplus}$: graded (formally add shifts, but consider only degree zero morphisms) additive (formally add sums), envelope.

3. $\mathcal{H} := \mathcal{H}_{\oplus}^{e}$ (Karoubi envelope).

<u>Main Theorem:</u> ① For any complete local ring $\mathbb{k}$ or field $\mathbb{k}$

$$\begin{array}{ccc} \{\text{indec. objects in } \mathcal{H}_{\mathbb{k}}\} / \text{iso. + shift} & \xrightarrow{\sim} & W \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ B_w & \longleftarrow & w \end{array}$$

② $[\mathcal{H}_{\mathbb{k}}] \xrightarrow{\sim} H$ Hecke algebra.

③ If $\mathbb{k} = \mathbb{Q}$: $[B_w] \longmapsto b_w$ (KL basis).

$Q = \operatorname{Frac} R$

④ $Q \otimes_R \mathcal{H}$ is semi-simple,

$$\begin{array}{ccc} [\mathcal{H}] & \xrightarrow{\sim} & H \\ \downarrow & & \downarrow{\scriptstyle v=1} \\ [Q \otimes_R \mathcal{H}] & \longrightarrow & \mathbb{Z}W \end{array}$$

Recall:

$\mathrm{Rep}_0 G \circlearrowleft \Theta_s$ wall crossing functor.

Conjecture (w/ Riche): $\mathcal{H}$ acts (on the right) on $\mathrm{Rep}_0 G$ via $B_s = \Theta_s$.

Recall also:

1. Tilt_0 is preserved by translation functors.
2. $T_{id} \cdot \Theta_s = 0$ for all $s \in S_f$.
3. $T_{id} \cdot \tilde{\alpha}_0 = T_{id} \cdot = 0$.

Hence if the conjecture holds then we get a functor of right $\mathcal{H}$-modules:

$$\Phi : \mathcal{H} / \langle B_s \mathcal{H} \mid s \in S_f \rangle \longrightarrow \mathrm{Tilt}_0^{\mathbb{F}_p}$$

$$\mathcal{H} / \langle B_s \mathcal{H} \mid s \in S_f \rangle \to \mathbb{F}_p \otimes_{k[\alpha_0]} \mathcal{AS} \nearrow \mathrm{Tilt}_0^{\mathbb{F}_p}$$

Thm: Conjecture implies Φ is an equivalence (need to kill $\tilde{\alpha}$).

Rmk: Grading on $\mathcal{AS}$ actually provides a grading on $\mathrm{Tilt}_0^{\mathbb{F}_p}$.

Consider the object N_{id}: the image of B_{id} in $\mathcal{AS}$.

$$\mathrm{End}_{\mathcal{H}}(B_{id}) = R \quad \boxed{f} \; .$$

But in $\mathcal{AS}$: $\alpha_s = {}^s = 0$, because it factors over B_s, for $s \in S_f$.

Thus we have a map: $k[\alpha_0] = R / \langle \alpha_s \mid s \in S_f \rangle \longrightarrow \mathrm{End}(N_{id})$.

~~Structure theorem for AS:~~

~~① $\mathrm{End}(N_{id}) = R/\langle \alpha_s \mid s \in S_f \rangle = \mathbb{h}[\tilde{\alpha}]$.~~

~~Need a little more detail on $\mathcal{H}$.~~

~~②~~

<u>Thm:</u> In $\mathbb{h}(\tilde{\alpha}) \otimes_{\mathbb{h}[\alpha]} \mathcal{A}S$ there exist objects Q_x, $x \in {}^f W$ with the following properties:

1. $Q_{id} = N_{id}$;
2. $Q_x f = x(f) Q_x \quad \forall f \in R$;
3. For all $x \in {}^I W$ we have, $Q_x B_s = \begin{cases} Q_x \oplus Q_{xs} & \text{if } xs \in {}^I W \\ 0 & \text{o/w}. \end{cases}$
4. $\mathrm{Hom}(Q_x, Q_y) = \delta_{x,y}\, \mathbb{h}(\tilde{\alpha}) \cdot id_{Q_x}$.
5. Q_x are the precisely the isomorphism classes of indec. objects in $\mathbb{h}(\tilde{\alpha}) \otimes_{\mathbb{h}[\alpha]} \mathcal{A}S$.

All easy and can be done "by heart".

This means that complicated calcs in $\mathcal{H}$ can be reduced to calcs in a polynomial ring in one variable.

Key point in proof: "infinite twist" $F^{\infty}_{w_f} \in K^{-}(\mathcal{H})$.

(idea came from link homology litterature.)