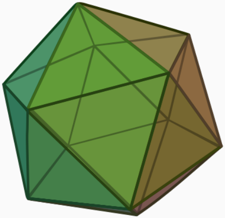
REPRESENTATION THEORY AND GEOMETRY

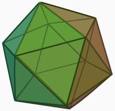
Geordie Williamson

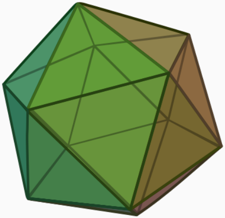
University of Sydney

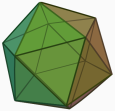
http://www.maths.usyd.edu.au/u/geordie/Heilbronn.pdf

REPRESENTATIONS



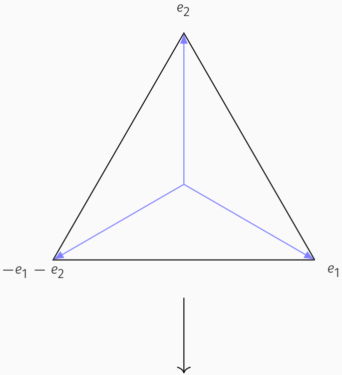
$\rightsquigarrow \quad \mathrm{Sym}\Big($  $\Big) \subset GL(\mathbb{R}^3).$



$\rightsquigarrow \quad \mathrm{Sym}\Bigg($  $\Bigg) \subset GL(\mathbb{R}^3).$

We obtain a representation of our group of symmetries

$$\rho : G \to GL(V).$$



$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

WHY STUDY REPRESENTATIONS?

Symmetric group S_n
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$$\{S_n\text{-sets}\}/\text{isomorphism} \xrightarrow{\sim} \{\text{subgroups of } S_n\}/\text{conjugation}$$

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Galois representations

The passage(s)

$$\{\text{varieties}/\mathbb{Q}\} \longrightarrow \{\text{Galois representations}\}$$

is one of the most powerful tools of modern number theory.

BURNSIDE'S OPINION

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

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BURNSIDE'S OPINION

PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter.

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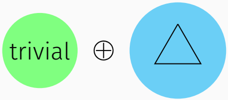

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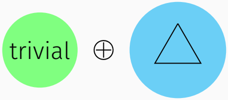

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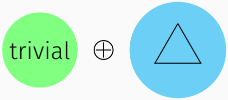

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A representation is semi-simple if it is isomorphic to a direct sum of simple representations.

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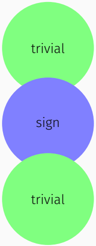
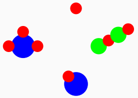


We write ("Grothendieck group", "multiplicities")

$$[\mathbb{F}_3^3] = [L] + [H/L] + [\mathbb{F}_3^3/H] = 2[\text{trivial}] + [\text{sign}].$$

SIMPLE REPRESENTATIONS

representations ↔ “matter”



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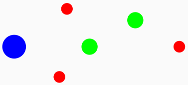

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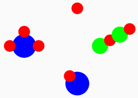

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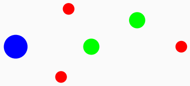

We search for a classification ("periodic table"), character formulas ("mass", "number of neutrons"), ...

PLAN OF TALK

"Semi-simple world"

- *Finite groups:*
  Maschke's theorem (1897).

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Related situations: non-compact Lie groups, p-adic groups...

COMMON FEATURES

representation theory

representation theory $\longleftrightarrow$ geometry

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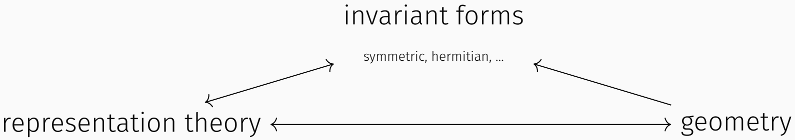
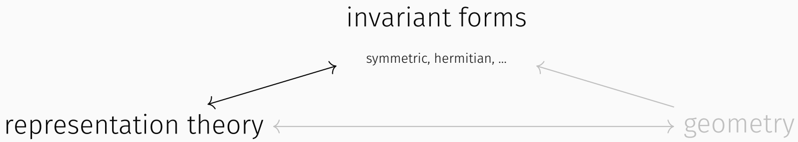
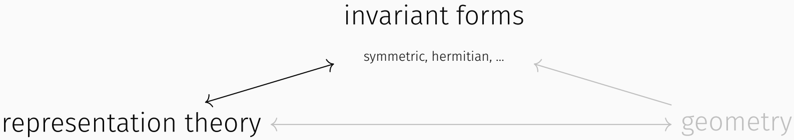
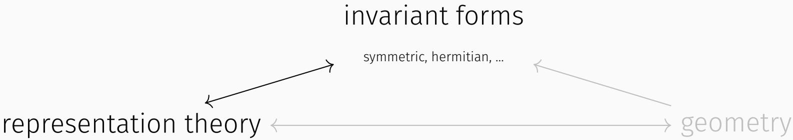



invariant forms
symmetric, hermitian, ...
representation theory
geometry



Related feature:
(hidden) semi-simplicity



Related feature:
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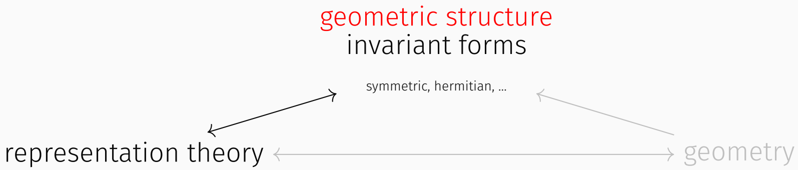
We do not assume that our forms are positive definite; signature plays an important role throughout.



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THE SEMI-SIMPLE WORLD

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Observation 2: Any representation of G admits a positive-definite geometric structure.

Take a positive-definite geometric structure $\langle -, - \rangle$ on V. Then

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum \langle gv, gw \rangle$$

defines a positive-definite and G-invariant geometric structure.

Example of "semi-simplicity via introduction of geometric structure".

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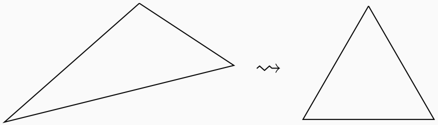
This is an example of "unicity of geometric structure".



Consider a compact Lie group K, e.g. S^1 or SU_2 or a finite group.

Weyl generalised these observations to K, with sum replaced by integral:

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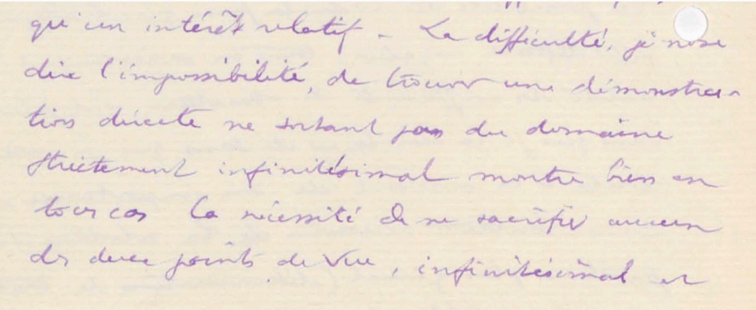
Existence and uniqueness of geometric structure still holds.

Élie Cartan to Hermann Weyl, 28 of March 1925:



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"…the difficulty, I dare not say the impossibility, of finding a proof which does not leave the strictly infinitesimal domain shows the necessity of not sacrificing either point of view …"

An algebraic ("infinitesimal") proof took 10 years, and involves the Casimir element (arises from an invariant form called the trace form).

"the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups". Hermann Weyl, 1949.

EXTENDED EXAMPLE: su_2 AND $\mathfrak{sl}_2$

$$SU_2 = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| AA^* = \mathrm{id}, \det A = 1 \right\} = \begin{matrix} \text{unit} \\ \text{quaternions.} \end{matrix}$$

$$\mathrm{Lie}(SU_2)_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C}) = \mathbb{C} \underset{\underset{f}{\parallel}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \oplus \mathbb{C} \underset{\underset{h}{\parallel}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \oplus \mathbb{C} \underset{\underset{e}{\parallel}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

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"I don't think it is the representations themselves, but the groups. I find SU_2, SL_2, S_n etc. amazing and beautiful animals (if I have a favourite, it is SU_2), but will probably never really understand them. I might someday understand their linear shadows though..."

– Quindici

SU_2 acts on its "natural representation":

$$\mathbb{C}^2 = \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix} = \mathbb{C}Y \oplus \mathbb{C}X.$$

For any $m \geqslant 0$, SU_2 acts naturally on homogenous polynomials in X, Y of degree m:

$$L_m := \mathbb{C}Y^m \oplus \mathbb{C}Y^{m-1}X \oplus \cdots \oplus \mathbb{C}Y^m X^{m-1} \oplus \mathbb{C}X^m.$$

The L_m for $m \geqslant 0$ are all irreducible representations of SU_2.

"spherical harmonics", "quantum mechanics".

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Action on L_m (here $m = 5$):

$\mathbb{C}Y^5$ $\quad$ $\mathbb{C}Y^4X^1$ $\quad$ $\mathbb{C}Y^3X^2$ $\quad$ $\mathbb{C}Y^2X^3$ $\quad$ $\mathbb{C}YX^4$ $\quad$ $\mathbb{C}X^5$

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Y^5	Y^4X^1	Y^3X^2	Y^2X^3	YX^4	X^5	
-5	-3	-1	1	3	5	h

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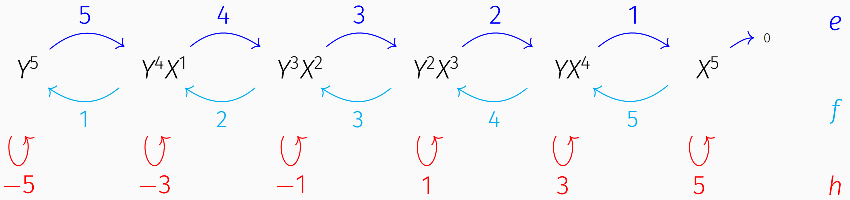

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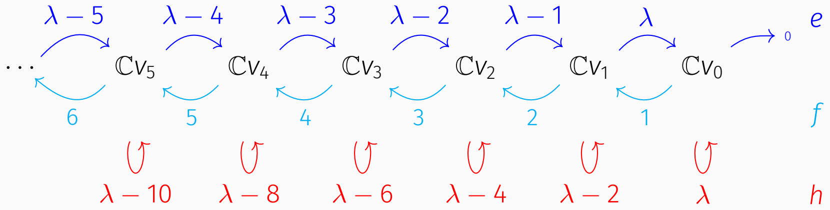
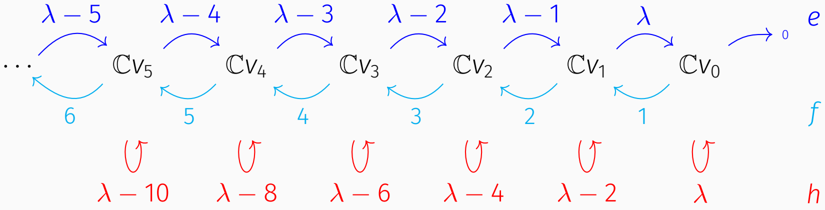
As vector spaces:

$$\Delta_\lambda = \bigoplus_{i \in \mathbb{Z}_{\geqslant 0}} \mathbb{C} v_i$$

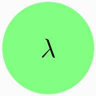
$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\underset{\substack{\parallel \\ f}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \oplus \mathbb{C}\underset{\substack{\parallel \\ h}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \oplus \mathbb{C}\underset{\substack{\parallel \\ e}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

The Verma module Δ_λ determined by $\lambda \in \mathbb{C}$:

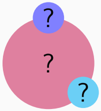




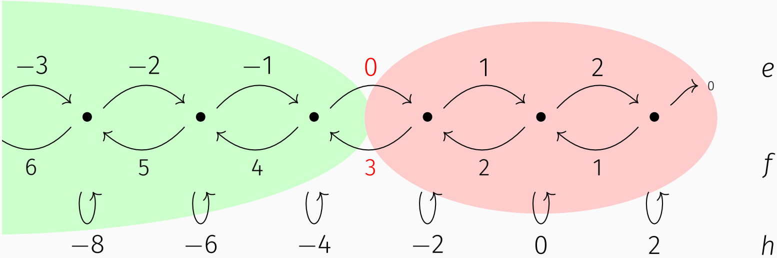
$\lambda \neq 0, 1, 2, 3, \ldots$: Δ_λ is simple, call it L_λ.



$\lambda = 0, 1, 2, 3, \ldots$: Δ_λ is not simple.

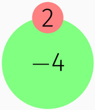


Example $\lambda = 2$



We have a subrepresentation isomorphic to Δ_{-4}, and

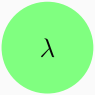
$$\Delta_2/\Delta_{-4} \cong L_2$$



(L_2 is our simple finite-dimensional representation from earlier.)

STRUCTURE OF VERMA MODULES

$\lambda \neq 0, 1, 2, 3, \ldots \quad : \quad \Delta_\lambda$ is simple and infinite-dimensional.



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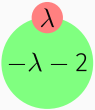

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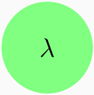

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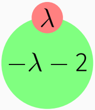

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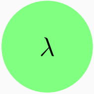

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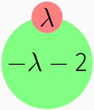

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(a) A single family of representations (Verma modules) produces all simple finite-dimensional representations.

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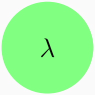

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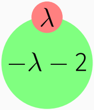

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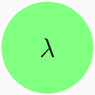

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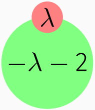

Summary:

(a) A single family of representations (Verma modules) produces all simple finite-dimensional representations.

(b) We get new infinite-dimensional simple representations.

(c) The structure of Verma modules varies (subtly) based on the parameter.

Kazhdan–Lusztig conjecture

$\mathfrak{g}$ is a complex semi-simple Lie algebra.

$\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.

W the Weyl group, which acts on $\mathfrak{h}$ as a reflection group.

Example

$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \{ n \times n$ matrices $X \mid \mathrm{tr} X = 0\}$.
$\mathfrak{h} =$ diagonal matrices $\subset \mathfrak{sl}_n(\mathbb{C})$
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Motivation

We think of the finite group W as being the skeleton of $\mathfrak{g}$.

We try to answer questions about $\mathfrak{g}$ in terms of W.

$\mathfrak{g}$ is a complex semi-simple Lie algebra.

"weight" $\lambda \in \mathfrak{h}^*$ $\rightsquigarrow$ *Verma module* Δ_λ.

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L_λ is called a simple highest weight module.

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Basic problem

Describe the structure of Δ_λ.
Which simple modules occur with which multiplicity?

Δ_λ : Verma module. L_λ : simple highest weight module.

Kazhdan-Lusztig conjecture (1979)

$$[\Delta_\lambda] = \sum_\mu P_{\lambda,\mu}(1)[L_\mu].$$

Here $P_{\lambda,\mu} \in \mathbb{Z}[v]$ is a Kazhdan-Lusztig polynomial.

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$$[\Delta_{x\cdot 0}] = \sum_{y \in W} P_{x\cdot 0, y\cdot 0}(1)[L_{y\cdot 0}].$$

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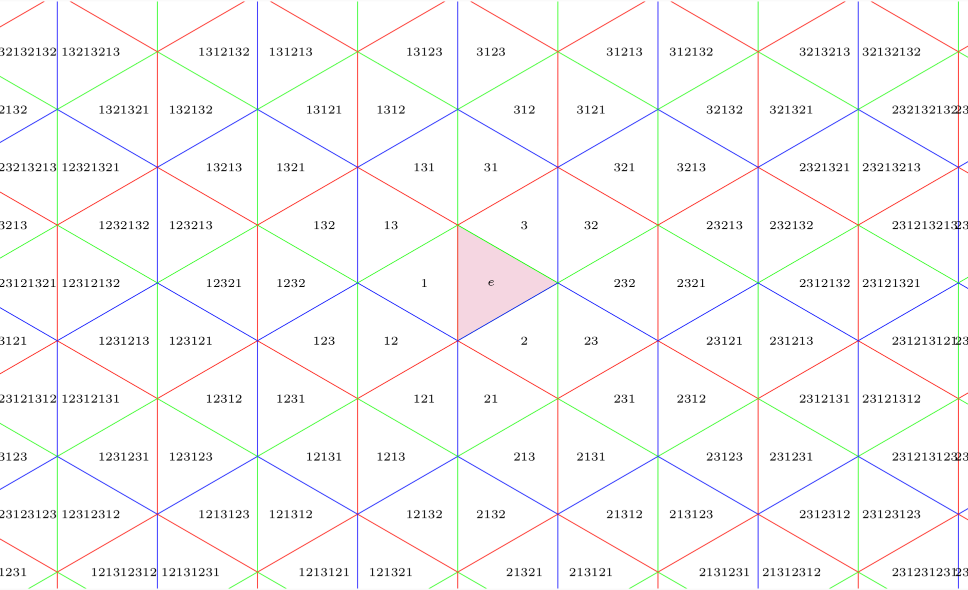
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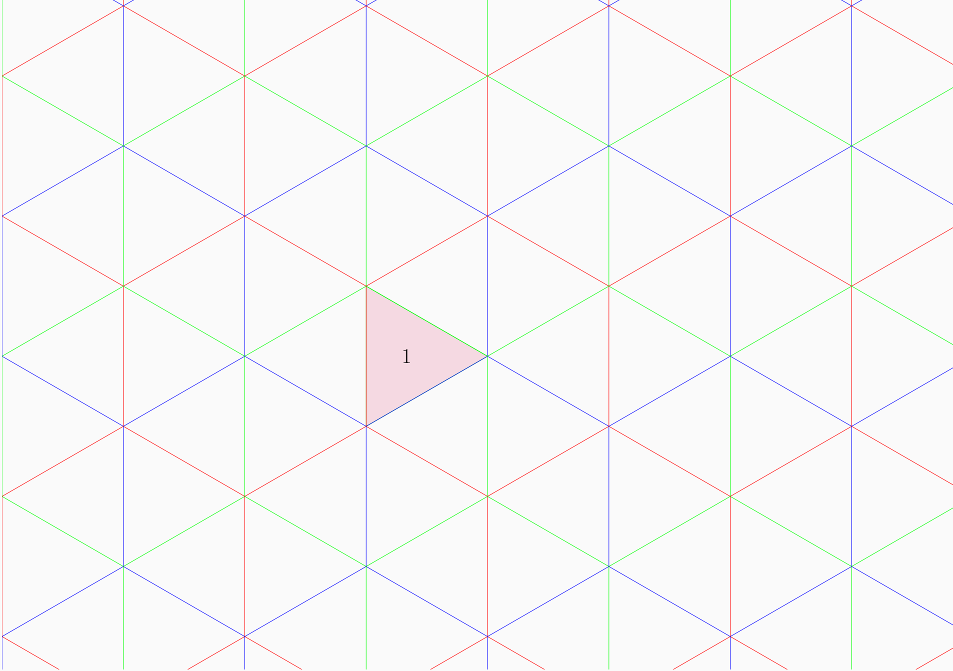
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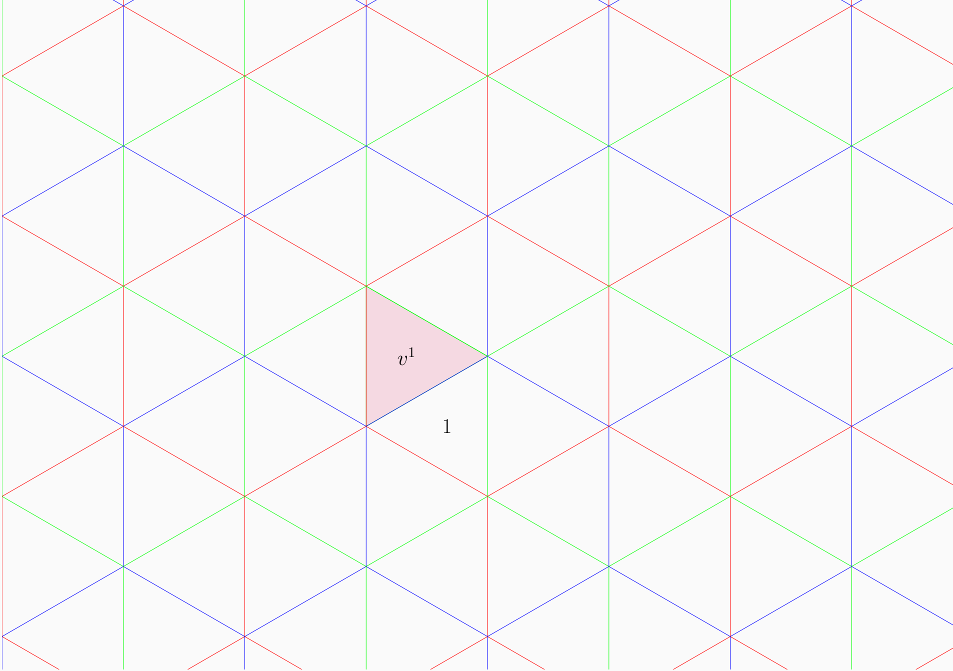
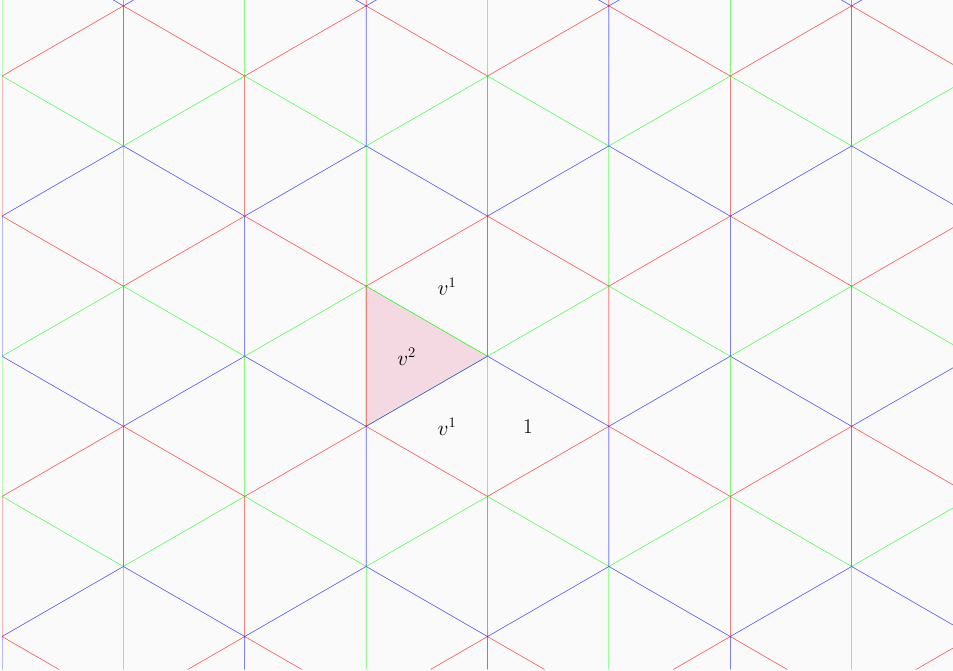
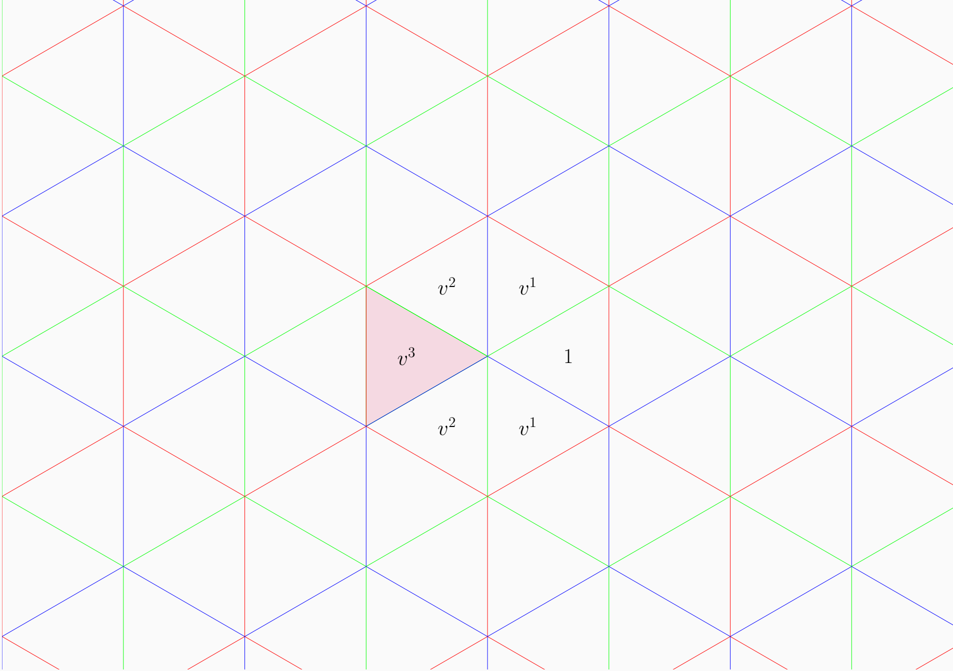
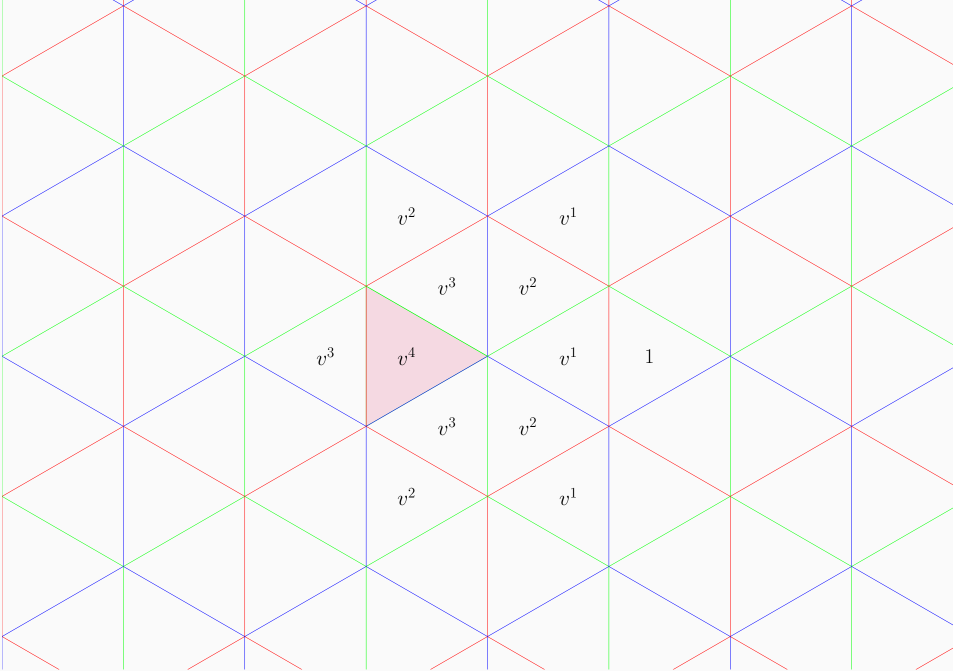
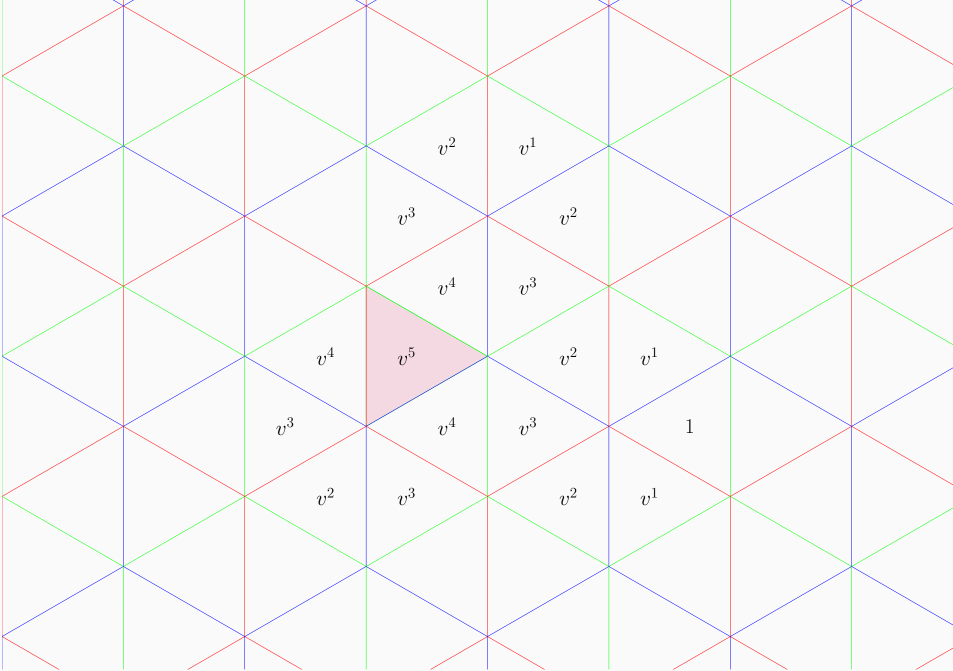

v^1
1


v^1
v^2
v^1
1

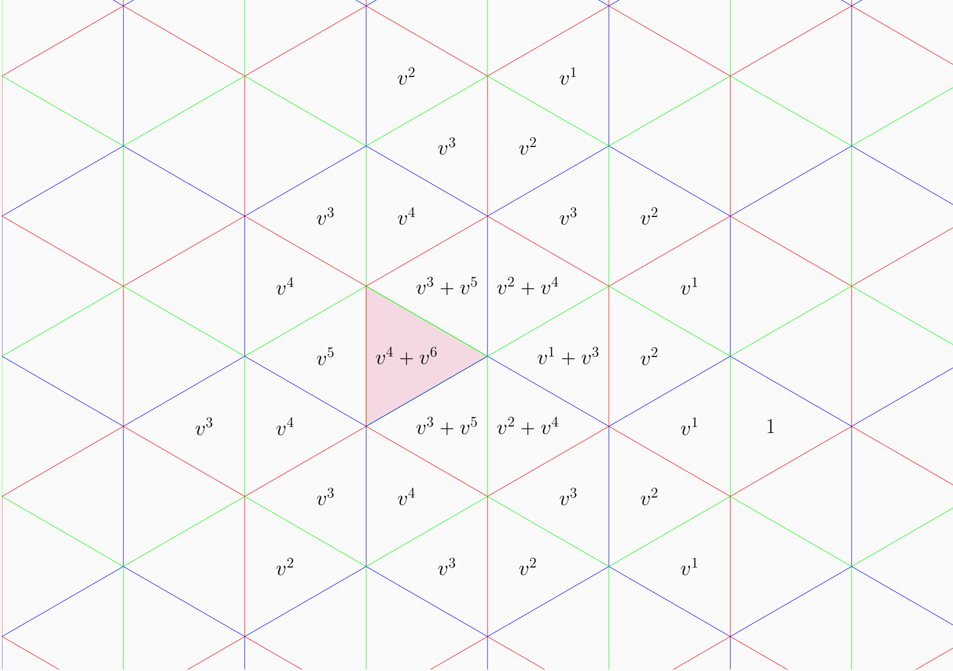

v^2
v^1
v^3
1
v^2
v^1



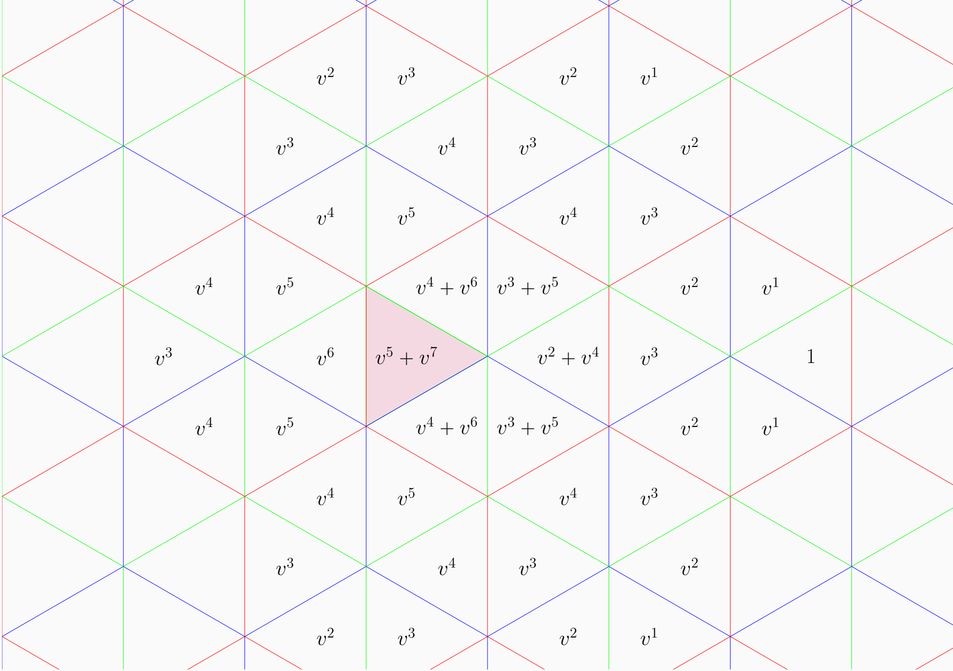
v^2
v^1
v^3
v^2
v^3
v^4
v^1
1
v^3
v^2
v^2
v^1



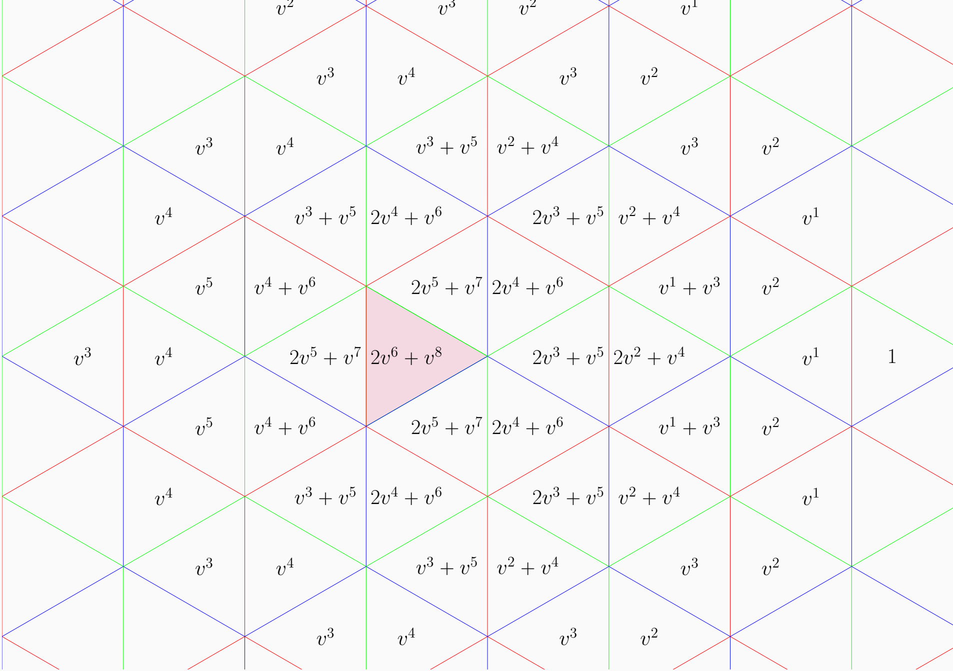
v^2 v^1
v^3 v^2
v^4 v^3
v^4 v^5 v^2 v^1
v^3 v^4 v^3 1
v^2 v^3 v^2 v^1



v^2
v^1
v^3
v^2
v^3
v^4
v^3
v^2
v^4
$v^3 + v^5$
$v^2 + v^4$
v^1
v^5
$v^4 + v^6$
$v^1 + v^3$
v^2
v^3
v^4
$v^3 + v^5$
$v^2 + v^4$
v^1
1
v^3
v^4
v^3
v^2
v^2
v^3
v^2
v^1



v^2 v^3 v^2 v^1
v^3 v^4 v^3 v^2
v^4 v^5 v^4 v^3
v^4 v^5 $v^4 + v^6$ $v^3 + v^5$ v^2 v^1
v^3 v^6 $v^5 + v^7$ $v^2 + v^4$ v^3 1
v^4 v^5 $v^4 + v^6$ $v^3 + v^5$ v^2 v^1
v^4 v^5 v^4 v^3
v^3 v^4 v^3 v^2
v^2 v^3 v^2 v^1


v^2 v^3 v^2 v^1
v^3 v^4 v^3 v^2
v^3 v^4 $v^3 + v^5$ $v^2 + v^4$ v^3 v^2
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v^5 $v^4 + v^6$ $2v^5 + v^7$ $2v^4 + v^6$ $v^1 + v^3$ v^2
v^3 v^4 $2v^5 + v^7$ $2v^6 + v^8$ $2v^3 + v^5$ $2v^2 + v^4$ v^1 1
v^5 $v^4 + v^6$ $2v^5 + v^7$ $2v^4 + v^6$ $v^1 + v^3$ v^2
v^4 $v^3 + v^5$ $2v^4 + v^6$ $2v^3 + v^5$ $v^2 + v^4$ v^1
v^3 v^4 $v^3 + v^5$ $v^2 + v^4$ v^3 v^2
v^3 v^4 v^3 v^2

JANTZEN CONJECTURE

A central theme of this talk is the omnipresence of forms in representation theory.

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I will now explain the relevance of the contravariant form to the Kazhdan-Lusztig conjecture.

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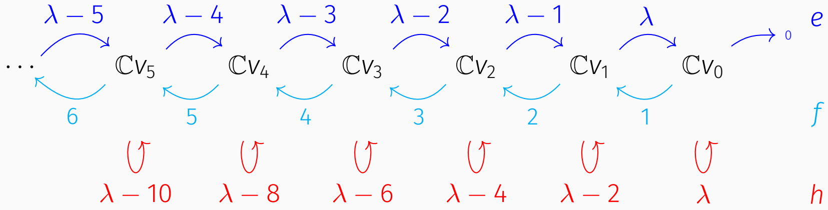
I will now explain the relevance of the contravariant form to the Kazhdan-Lusztig conjecture.

This will end up explaining why there is a q in the Kazhdan-Lusztig conjecture.

THE CONTRAVARIANT FORM

Our Verma module for $\mathfrak{sl}_2$ from earlier:



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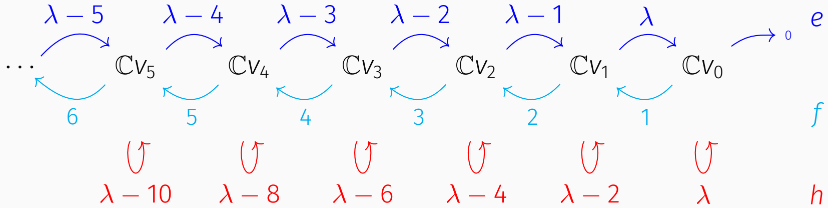
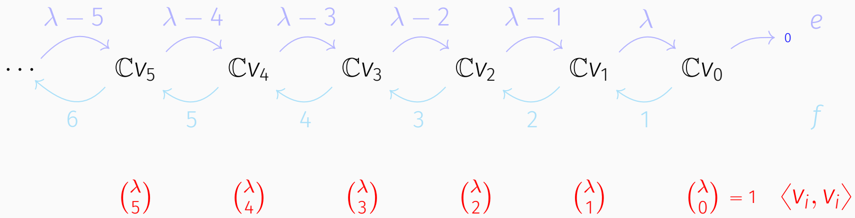
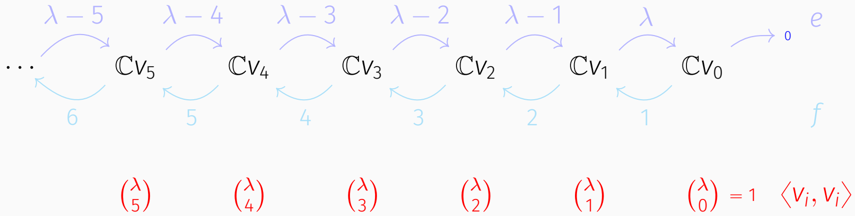


There is a unique symmetric form (the contravariant form) satisfying

$$\langle v_0, v_0 \rangle = 1,$$
$$\langle hv, v' \rangle = \langle v, hv' \rangle \quad \text{and} \quad \langle ev, v' \rangle = \langle v, fv' \rangle.$$


λ − 5
λ − 4
λ − 3
λ − 2
λ − 1
λ
e
0
...
ℂv5
ℂv4
ℂv3
ℂv2
ℂv1
ℂv0
6
5
4
3
2
1
f
(λ 5)
(λ 4)
(λ 3)
(λ 2)
(λ 1)
(λ 0) = 1
⟨vi, vi⟩

THE CONTRAVARIANT FORM



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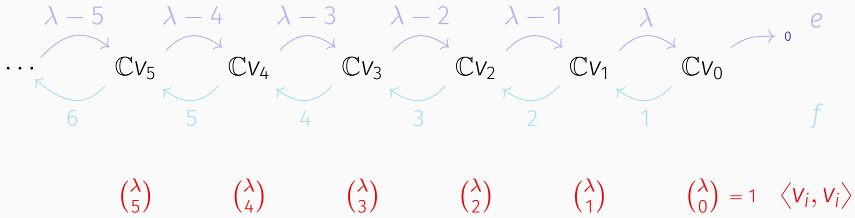
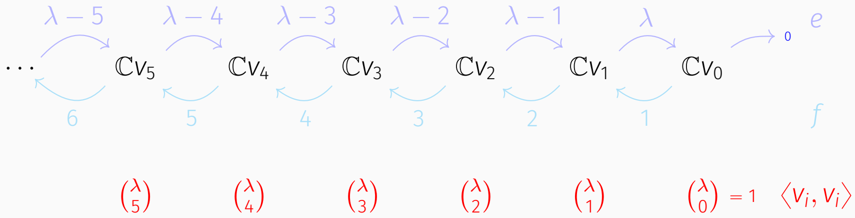


The contravariant form is non-degenerate if and only if $\binom{\lambda}{i} \neq 0$ for all $i \geqslant 0$, which is the case if and only if is not a non-negative integer.



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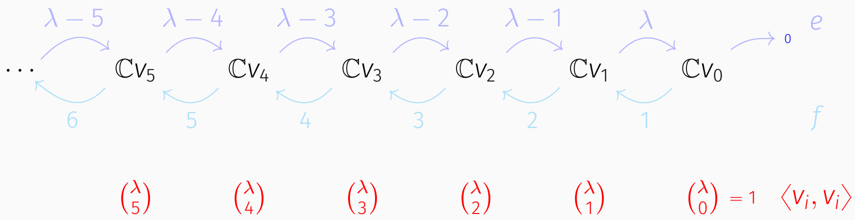
We have seen that this is precisely the condition for Δ_λ to be irreducible.

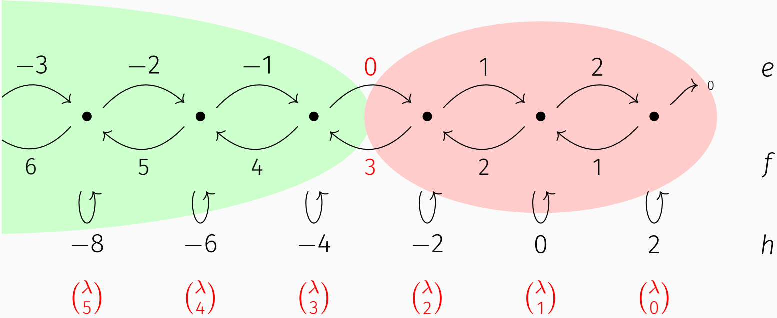
VANISHING OF THE CONTRAVARIANT FORM



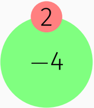
Jantzen's beautiful idea

View the form $\langle -, - \rangle$ as a family of forms depending on the parameter λ. Then study the vanishing behaviour of this form.

Example $\lambda = 2$



At $\lambda = 2$, the contravariant form vanishes to order 1 precisely on the subrepresentation Δ_{-4} identified earlier.



Verma modules and their contravariant forms give a family over $\mathfrak{h}^*$.

$\rightsquigarrow$ Jantzen filtration via order of vanishing of contravariant form:

$$\Delta_\lambda \supset F^1\Delta_\lambda \supset F^2\Delta_\lambda \supset \dots$$

Subquotients $F^i\Delta_\lambda/F^{i+1}\Delta_\lambda$ admit non-degenerate forms.

Jantzen conjecture (1979)

The subquotients $F^i\Delta_\lambda/F^{i+1}\Delta_\lambda$ are semi-simple.

"although Verma modules are not semi-simple, the layers of the Jantzen filtration do not interact"

Jantzen conjecture (1979)

The subquotients $F^i\Delta_\lambda/F^{i+1}\Delta_\lambda$ are semi-simple.

- Explains why Kazhdan-Lusztig polynomials are polynomials:

$$(\Delta_\lambda : L_\mu) = P_{\lambda,\mu}(1) \qquad \rightsquigarrow \qquad \sum_i (\mathrm{gr}^i_F \Delta_\lambda : L_\mu) q^i = P_{\lambda,\mu}$$

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- Implies Kazhdan-Lusztig conjecture (Gabber-Joseph).
- Instance of "hidden semi-simplicity".

Jantzen conjecture (1979)

The subquotients $F^i\Delta_\lambda/F^{i+1}\Delta_\lambda$ are semi-simple.

- Explains why Kazhdan-Lusztig polynomials are polynomials:

$$(\Delta_\lambda : L_\mu) = P_{\lambda,\mu}(1) \qquad \rightsquigarrow \qquad \sum_i (\mathrm{gr}^i_F \Delta_\lambda : L_\mu) q^i = P_{\lambda,\mu}$$

- Implies Kazhdan-Lusztig conjecture (Gabber-Joseph).
- Instance of "hidden semi-simplicity".
- Important in unitarity problem for real Lie groups (signature).

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Geometric proofs: D-modules, perverse sheaves, weights...

Algebraic proofs: "shadows of Hodge theory",
i.e. invariant forms ("geometric structures")
still satisfying Poincaré duality, Hard Lefschetz, Hodge-Riemann